APPLICATION OF EXPONENTIAL GROWTH AND DECAY

There are many real-world applications of situations that can be modeled by exponential functions. In this section we will look at two of these.

Example 1 DOUBLING TIME

Joe buys a comic book in 1980 for $1.50. He keeps it in its protective covering and monitors the value of the comic book over the years. Below is a table that gives the value of the comic book at the end of the first 10 years.

Year	0	1	2	3	4	5	6	7	8	9	10
Value	$1.50	$1.73	$1.98	$2.28	$2.62	$3.02	$3.47	$3.99	$4.59	$5.28	$6.07

This is an exponential function and we can find the common ratio, *r*, as always, by dividing any term by the term before it (since we are given successive terms!).

Thus: $r = \frac{1.73}{1.50} = \frac{1.98}{1.73} = \frac{2.28}{1.98} = \ldots \approx 1.15$

As we have found from the examples in the last section, the equation that gives this value, then, is: $V = 1.50 \cdot 1.15^x$.

a) How long does it take the value to double?

From the data, we can see that the value of the comic book doubles from the original value of $1.50 to approximately $3.00 in 5 years. Thus, we say the **doubling time for the value of this comic book is 5 years.**

b) **Find another equation that could be used to model this data.**

We can tell from the data that the y-values **double** every **5** years. This leads us to a second equation that represents the data: $\boxed{V = 1.50 \cdot 2^{\frac{x}{5}}}$

When $x = 0$ years, $V = 1.50(1) = 1.50$

$x = 5$ years, $V = 1.50(2) = 3.00$

This equation works for these two values! Check for other values of x and you will see that this equation also gives the values in the table. The values are actually approximate values and there may be a slight discrepancy due to rounding.

$x = 1$ year, $V = 1.50(1.15) = 1.72$

$x = 7$ years, $V = 1.50(2.64) = 3.96$

We have given two equations for the **same** set of data, $y = 1.50 \cdot (1.15)^x$ and $y = 1.50 \cdot 2^{\frac{x}{5}}$. Thus, it follows that these equations are equivalent to each other. If we look at these two equations, we can see that they are indeed equivalent. They each have a common factor of 1.50, so this means that 1.15^x must equal $2^{\frac{x}{5}}$. Now, $2^{\frac{x}{5}} = (2^{\frac{1}{5}})^x$ and $2^{\frac{1}{5}} = 1.15$. Thus, we see that $2^{\frac{x}{5}} = 1.15^x$, which shows that the equations are, in fact, equivalent.

So, given that the original value is 1.5,

- if we know that the value **doubles** in **5** years, the equation is: $V = 1.5 \cdot 2^{\frac{x}{5}}$.
- if we know that the value **doubles** in **11** years, the equation is: $V = 1.5 \cdot 2^{\frac{x}{11}}$.
- if we know that the value **triples** in **7** years, the equation is: $V = 1.5 \cdot 3^{\frac{x}{7}}$.

- if we know that the value **quadruples in 15** years, the equation is: $V = 1.5 \cdot 4^{\frac{x}{15}}$.
- if we know that the value **increases by a factor of 3.56** in **2** years, the equation is:

  $$V = 1.5 \cdot (3.56)^{\frac{x}{2}}.$$

c) Using the idea from part (b) above, find from the table when the value of the comic book will quadruple and then find the equation that uses this quadrupling time.

From the table, the value **quadruples** from $1.50 to $6.00 in approximately **10 years**. We can state that the equation that gives the value as: $\mathbf{V = 1.50 \cdot 4^{\frac{x}{10}}}$.

You can check this formula for various values of x years and you will see that you will obtain the approximate table entries for the value of the comic book.

Example 2

Anita purchased a book for $13.50 in 1990. If the value of the book **doubled** every 7 years, how much would it be worth in 4 years, 11 years, 50 years?

Solution:

Since it states the value is doubled we can write the equation as: $V = 13.50 \cdot 2^{\frac{x}{7}}$.

So: after 4 years $V = 13.50 \cdot 2^{\frac{4}{7}} = \$20.06$

after 11 years $V = 13.50 \cdot 2^{\frac{11}{7}} = \$40.12$

after 50 years $V = 13.50 \cdot 2^{\frac{50}{7}} = \$1907.86$

Example 3

A culture is found to have 2300 bacteria. The number of bacteria **triples** in 4 h. Find the amount of bacteria at the end of one day.

Solution

The equation for this will be: $A = 2300 \cdot 3^{\frac{x}{4}}$, where x is the # of hours. We use a base of 3 since we are given the tripling time.

So: In 24 hours: $A = 2300 \cdot 3^{\frac{24}{4}} = 1676700$ bacteria.

The three examples above are each exponential functions that exhibit **exponential growth.** We now look at some applications of exponential functions as they relate to **exponential decay.**

Example 4 HALF-LIFE

Consider the following data:

x	0	3	6	9	12
y	48	24	12	6	3

It is obvious that the function is decreasing. We can also see that the y-values are halved every 3 years. We say that the **half-life of y is 3 years.**

Using theory similar to the past examples, we can write the equation of the exponential function that represents this data.

$y = 48 \cdot \left(\frac{1}{2}\right)^{\frac{x}{3}}$ or $y = 48 \cdot (0.5)^{\frac{x}{3}}$ The base is now $\frac{1}{2}$ since we are talking about half-life and the denominator of the fraction in the exponent is 3 since this is the time required for y to be halved.

We could now use this equation to find the value of y at any time.