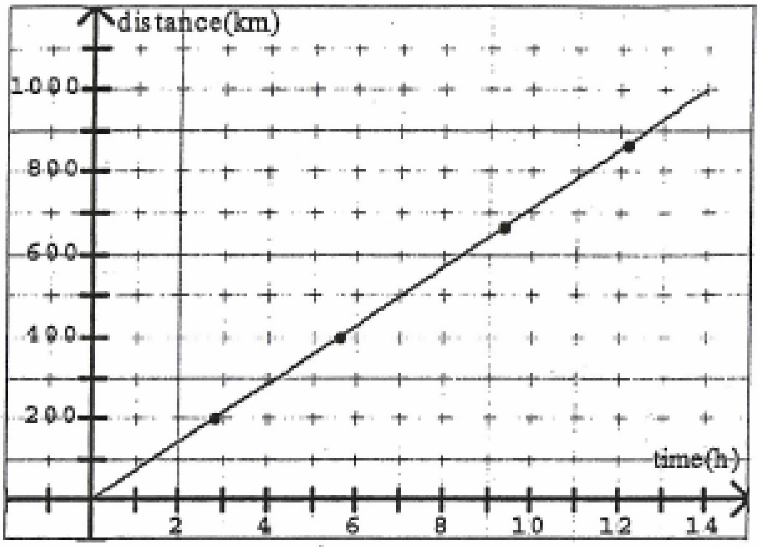
GRAPHICAL REPRESENTATION OF RATE OF CHANGE

The following data and graph depicts the distance a car travels over a period of 14 hours. As always, the independent variable, which in this case is the time in hours, is on the x-axis, while the dependent variable, distance in km, is on the y-axis.

time (h)	distance (km)
0.00	0.00
0.93	66.67
1.87	133.33
2.80	200.00
3.73	266.67
4.67	333.33
5.60	400.00
6.53	466.67
7.47	533.33
8.40	600.00
9.33	666.67
10.27	733.33
11.20	800.00
12.13	866.67
13.07	933.33
14.00	1000.00



We can see that when the data is plotted, the result is a **straight line**. Four points are marked: (2.80, 200.00) , (5.60, 400.00) , (9.33, 666.67) , and (12.13, 866.67).

These points can be used to find the average rate of change of distance with respect to time.

Choose any two points on the graph, say (2.80,200.00) and (9.33, 666.67). Using the steps in the earlier examples, we can find the average rate of change.

average rate of change $= \frac{(666.67 - 200.00)km}{(9.33 - 2.80)h} = \frac{466.67km}{6.53h} = 71.5\,km/h$

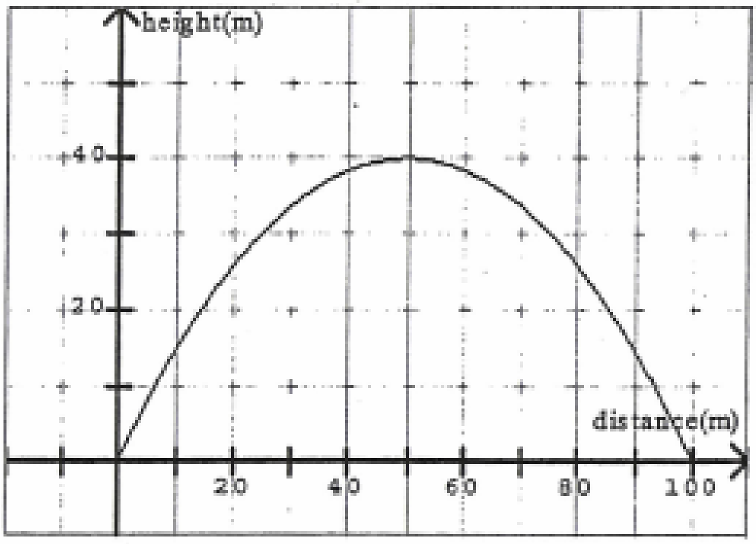
If we use any two other points on the graph, say, (5.60, 400.00) and (12.13, 866.67).

average rate of change $= \frac{(866.67 - 400.00)km}{(12.13 - 5.60)h} = \frac{466.67km}{6.53h} = 71.5\,km/h$

The average rate of change was the same value even though we took two different points each time. This is, of course, because the graph is linear. **For a linear function, the average rate of change will be unchanged regardless of which two points are chosen.**

So, what happens if the graph is not linear? Consider the following data and graph that depicts a baseball thrown from the outfield to home plate. For simplicity, we will assume that the outfielder releases it at a height of 0 m and the catcher catches it at a height of 0 m. The data shows the height of the ball vs the horizontal distance it has traveled.

distance (m)	height (m)
0.00	0.00
9.09	13.22
18.18	23.80
27.27	31.74
36.36	37.02
45.45	39.67
50.00	40.00
54.55	39.67
63.64	37.02
72.73	31.74
81.82	23.80
90.91	13.22
100.00	0.00



The plot of the graph is obviously not linear. It is, in fact, **quadratic**. It should be evident that the average rate of change of the height with respect to the horizontal distance is **not constant** in this instance. Gravity slows the vertical climb of the ball until eventually it changes direction and begins to fall back towards the earth.

Although the average rate of change is continually changing for this function, it is still possible to calculate the average rate of change **over a certain interval**.

Example:

Find the average rate of change of the height of the ball with respect to the horizontal distance as the horizontal distance changes from

a) 63.64 m to100 m

b) 9.09 m to 36.36 m

c) 45.45 m to 54.55 m

Solution:

a) average rate of change $= \dfrac{(0-37.02)m}{(100-63.64)m} = \dfrac{-37.02m}{36.36m} = -1.02m/m$

For this interval, the vertical distance **decreases** (the negative sign indicates this) 1.02 metres in height for every metre the ball travels horizontally, on average.

b) average rate of change $= \dfrac{(37.02-13.22)m}{(36.36-9.09)m} = \dfrac{23.80m}{27.27m} = .87\,m/m$

For this interval, the vertical distance **increases** (the positive sign indicates this) .87 metres in height for every metre the ball travels horizontally, on average.

c) average rate of change $= \dfrac{(39.67-39.67)m}{(54.55-45.45)m} = \dfrac{0\,m}{9.1m} = 0\,m/m$

For this interval, the **average height** does not increase nor decrease.

RATE OF CHANGE AS SLOPE

From our discussions of the past two examples, it is easy to relate the concept of **average rate of change** to that of the **slope of a line.**

Recall that : $$\text{slope} = m = \frac{\textit{change in } y}{\textit{change in } x} = \frac{\Delta y}{\Delta x}$$

From the first example in this section, the linear plot, what we actually did was find the slope of the line segment, using various points in our calculations. We know from past work that the slope of any straight line is constant. The values of the rates of change we found for the linear plot were also constant. **Thus, the average rate of change of a linear function is simply the slope of the line.**

The slope of the line, and thus, the rate of change will be

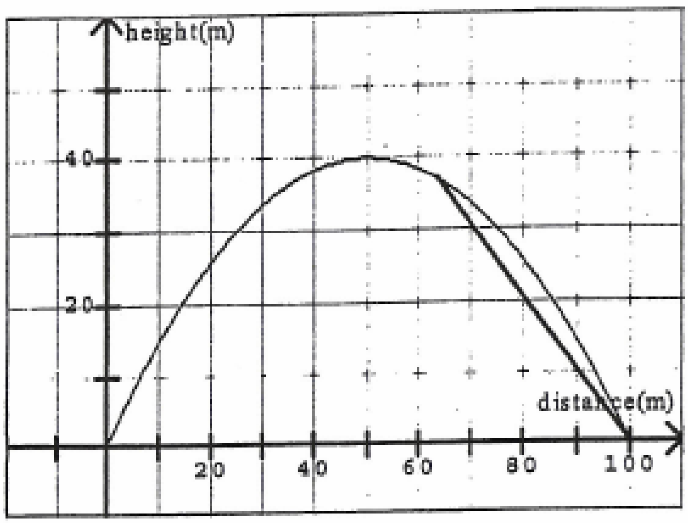
a) positive if the line rises from left to right.

b) negative if the line falls from left to right.

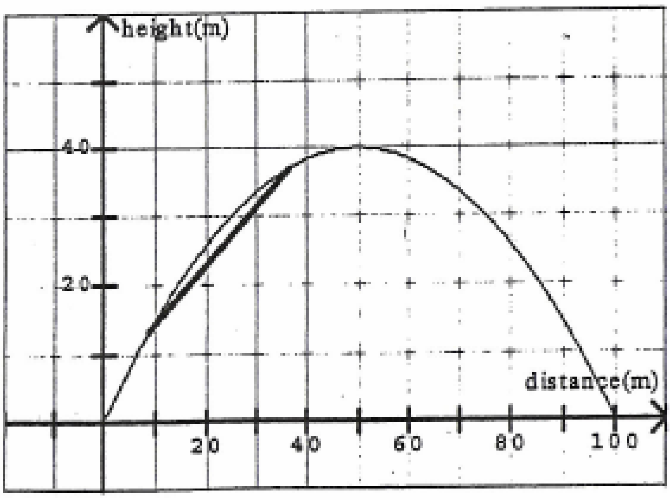
c) zero if the line is horizontal.

For the nonlinear plot, it was found that the rate of change was **not constant**. It changed depending on the interval we chose. Sometimes the rate was positive, sometimes negative, sometimes zero. Again, this is related to the slope of a line. To illustrate this, we will re-visit the last example where the baseball was thrown from the outfield to home plate.

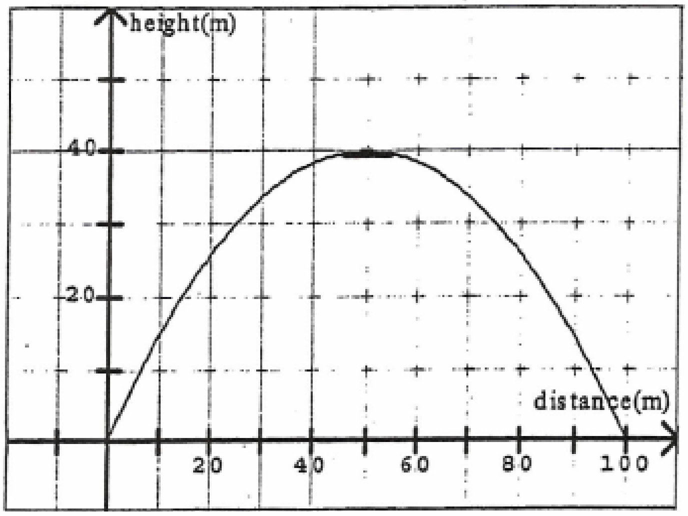
What we will do is draw a line through various pairs of points on our graph. A line joining two points is called a **secant** line. We will then calculate the slope of this secant line which will give us the rate of change of the height with respect to horizontal distance for that interval.



The line drawn joining the two points is called a **secant** line. It can be seen that the secant line falls from left to right. Thus, we would suspect the slope of this line to be negative. The average rate of change we found was -1.02. **This value is, in fact, the slope of the secant line joining the two points of our chosen interval.**



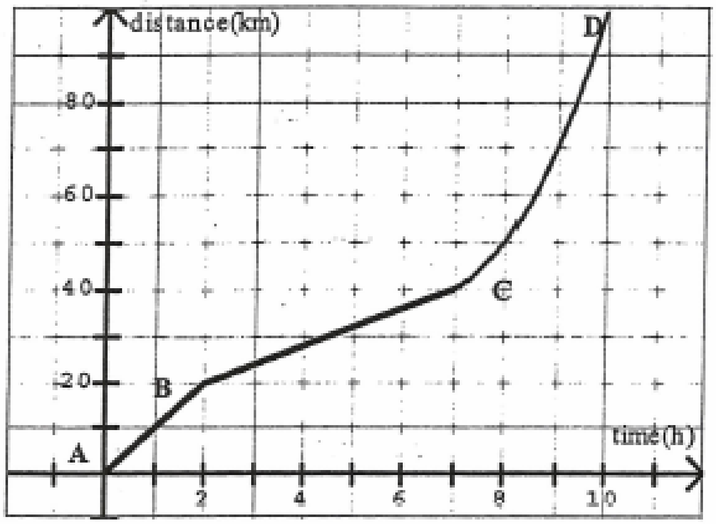
For the second part of the example, it can be seen that the secant line rises from left to right. Thus, we would suspect the slope of this line to be positive. The average rate of change we found was + .87. **This value is, in fact, the slope of the secant line joining the two points of our chosen interval.**



For the third part of the example, it can be seen that the secant line is horizontal. Thus, we would suspect the slope of this line to be zero. The average rate of change we found was 0. **This value is, in fact, the slope of the secant line joining the two points of our chosen interval.**

The conclusion that we can draw is that the average rate of change over a certain interval is equal to the slope of the secant line joining the endpoints of the chosen interval. If the function is linear, the average rate of change will be constant throughout, no matter what interval we choose. If the function is nonlinear, the average rate of change will change from interval to interval.

EXAMPLE 1: The graph below shows distance versus time for a bus trip.



Analyzing the graph tells us that there are three distinct parts of the trip.

1. The bus starts and travels 2 h at a constant speed (A to B is linear),
2. then travels 5 h at a different constant speed (B to C is also linear),
3. and finally travels the last 3 h at a non-constant speed (C to D is nonlinear).

We can find the average rate of change of distance to time(average speed) for each part of the trip.

A to B:

average speed $= \dfrac{(20-0)km}{(2-0)h} = 10\,km/h$ **This will be constant throughout the interval.**

B to C:

average speed $= \dfrac{(40-20)km}{(7-2)h} = 4\,km/h$ **This will be constant throughout the interval.**

C to D:

Since the graph is not linear here, we will not get a constant value for the average speed. It depends upon which points we choose to use from the graph. We will use the complete interval from 7 h to 10 h.

average speed $= \dfrac{(100-40)km}{(10-7)h} = 20\,km/h.$

The average speed will vary throughout the interval. Sometimes it will be less than 20 km/h, sometimes more than 20 km/h.