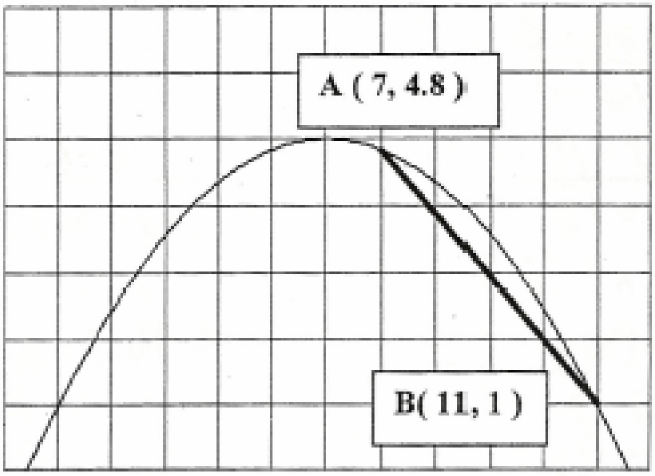
INSTANTANEOUS RATE OF CHANGE

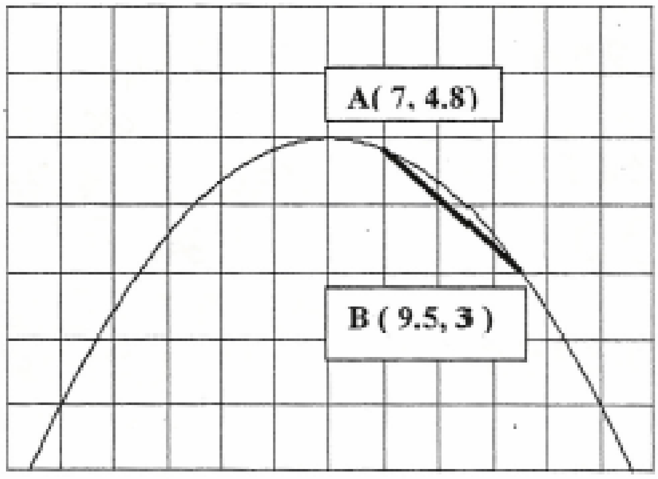
In the previous section, we were able to relate the average rate of change to the concept of slope. The process involved taking two points on the graph, drawing the secant line that connected those two points, and finding the slope of this secant line. The value obtained was the average rate in the interval between the two points we chose.



The slope can be calculated as:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{1 - 4.8}{11 - 7} = \frac{-3.8}{3} = -1.26$$

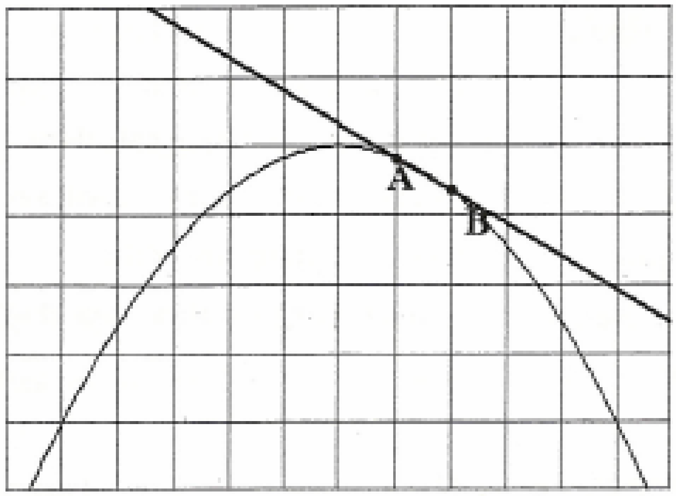
What will happen if we keep **A** as one of our points and move **B** a little closer to **A?**



The slope can be calculated as

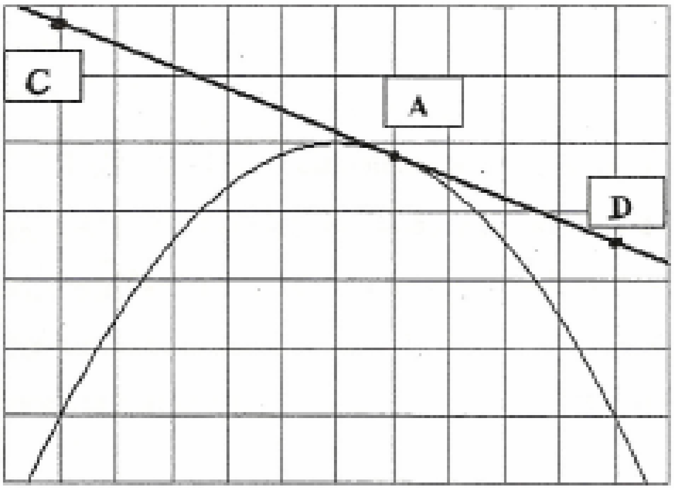
$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3 - 4.8}{9.5 - 7} = \frac{-1.8}{2.5} = -0.72$$

The slope between **A** and **B** has changed.



Here with the points even closer the slope can be calculated as m = -.48.

If we continue this a few more times, each time making point **B** closer and closer to **A**, eventually we will get a line that actually touches the graph at one point only, at point **A**. **This line that touches a graph at one point only is called the <u>tangent line to point A</u>.** Further, **the slope of this tangent line at A gives the <u>instantaneous rate of change at point A</u>.** It is no longer an average but the actual rate of change at that point.



To find the slope of a tangent, choose two points on the tangent as far apart as possible. For this example we will use points C(1, 6.8) and D(11, 3.6). Finding the slope between these two points will give us the actual rate of change at point **A**.

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3.6 - 6.8}{11 - 1} = \frac{-3.2}{10} = -0.32$$

The diagram above shows the **tangent line** through the point A. Its slope is measured to be - .32. Thus the **instantaneous rate of change at point A is exactly -.32**. You will not be required to determine the slope of a tangent however it will be helpful if you understand the process used to calculate the slope of a tangent to a curve.

EXAMPLE 1: FINDING THE QUADRATIC EQUATION FROM ORDERED PAIRS

The following table gives the height of a projectile at three different times after it is shot from ground level. The data can be used to find the quadratic equation that describes this motion.

Time (s)	Height (m)
1	44.10
3	102.90
8	78.40

a) Find the quadratic equation that fits this data.

b) Use this equation to find the maximum height of the projectile, the time at which it reaches maximum height, and the time at which it strikes the ground.

c) What is the instantaneous rate of change of the height at 2 s? at 5 s? at 9s?

The equation then, is $h = -4.9t^2 + 49t$

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b) **Find vertex of this quadratic function.**

$$h = -4.9t^2 + 49t$$
$$h = -4.9(t^2 - 10t)$$
$$h - 122.5 = -4.9(t^2 - 10t + 25)$$
$$h - 122.5 = -4.9(t - 5)^2$$
$$h = -4.9(t - 5)^2 + 122.5$$

Thus, the maximum height reached is 122.5 m which occurs at 5 s.

Find roots of the corresponding quadratic equation.

$-4.9t^2 + 49t = 0$

$$t = \frac{-49 \pm \sqrt{(49)^2 - 4(-4.9)(0)}}{-9.8} = 0 \text{ s and } 10 \text{ s.}$$

The projectile hits the ground after 10 s.

Notice that we could have predicted that the time to reach ground would be 10 s, which is double the time at maximum height. This happens since the ball started at the ground (There is no constant value in the quadratic ; c = 0!) This results in the graph being symmetrical from t = 0 s to t = 10 s and the maximum height will occur at the midway point in time (5 s). We can't use this line of thinking if the projectile does not start at ground level, as in the other examples.

c) The instantaneous rate of change at 2 s is found, as always, by finding the height 1.9 s and 2.1 s.

When t = 1.9 s:

$h = -4.9(1.9)^2 + 49(1.9)$
$h = -4.9(3.61) + 93.1$
$h = -17.689 + 93.1$
$h = 75.411$ m

When t = 2.1 s:

$h = -4.9(2.1)^2 + 49(2.1)$
$h = -4.9(4.41) + 102.9$
$h = -21.609 + 102.9$
$h = 81.291$ m

$$\text{IROC} = \frac{(81.291 - 75.411)\text{m}}{(2.1 - 1.9)\text{s}} = \frac{5.88 \text{ m}}{0.2 \text{ s}} = 29.4 \text{ m/s}$$

At 5 seconds...

When t = 4.9 s:

$h = -4.9(4.9)^2 + 49(4.9)$
$h = -4.9(24.01) + 240.1$
$h = -117.649 + 240.1$
$h = 122.451$ m

When t = 5.1 s:

$h = -4.9(5.1)^2 + 49(5.1)$
$h = -4.9(26.01) + 249.9$
$h = -127.449 + 249.9$
$h = 122.451$ m

$$\text{IROC} = \frac{(122.451 - 122.451)\text{m}}{(5.1 - 4.9)\text{s}} = \frac{0 \text{ m}}{0.2 \text{ s}} = 0 \text{ m/s}$$

At 9 seconds...

When t = 8.9 s:

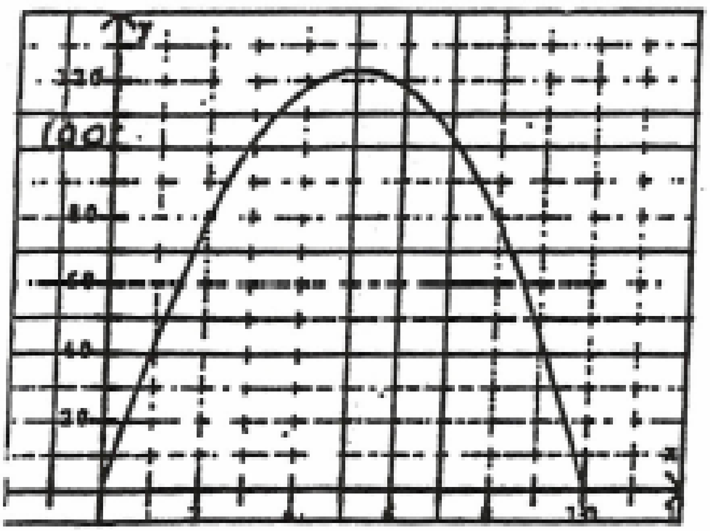
$h = -4.9(8.9)^2 + 49(8.9)$
$h = -4.9(79.21) + 436.1$
$h = -388.129 + 436.1$
$h = 47.971$ m

When t = 9.1 s:

$h = -4.9(9.1)^2 + 49(9.1)$
$h = -4.9(82.81) + 445.9$
$h = -405.769 + 445.9$
$h = 40.131$ m

$$\text{IROC} = \frac{(40.131 - 47.971)\text{m}}{(9.1 - 8.9)\text{s}}$$
$$= \frac{-7.84 \text{ m}}{0.2 \text{ s}}$$
$$= -39.2 \text{ m/s}$$

The graph of the function looks like this.



Notice that the signs of the rates of change we found above at 2s, 5s, and 9s, correspond with the slopes of the tangents at these points on the graph (as we knew would happen).

EXAMPLE 2: ANOTHER QUADRATIC FUNCTION

A batter in baseball makes contact with the ball at a height of 1m and hits it at a speed of 29.4 m/s.

a) State the equation for the height, h, in seconds.

b) Determine the average rate of change in the height of the ball during the time interval from 2 to 2.5 seconds.

c) Determine the maximum height of the ball and the time at what time it reaches maximum height.

d) How long did the ball remain in the air?

e) Find the velocity of the ball after 2 s.

SOLUTION:

a) The equation: $h = -4.9t^2 + 29.4t + 1$

b) $h(2.5) = 43.875$ m

$h(2) = 40.2$ m

$$\text{average rate of change} = \frac{(43.875 - 40.2)m}{(2.5 - 2)s} = 7.35 m/s$$

c) To find maximum height, complete the square to find the vertex.

$$h - 1 = -4.9t^2 + 29.4t + 1$$

$$h - 1 = -4.9\,(t^2 - 6t)$$

$$h - 1 - 44.1 = -4.9\,(t^2 - 6t + 9)$$

$$h - 45.1 = -4.9\,(t - 3)^2$$

$$h = -4.9\,(t - 3)^2 + 45.1$$

The vertex of this parabola is (3, 45.1). This tells us that the maximum height reached by the ball is 45.1 m and it happens at 3 seconds after impact.

d) To find the total time of flight, we can solve the corresponding quadratic equation when $h = 0$.

$$0 = -4.9t^2 + 29.4t + 1$$

Using the quadratic formula yields $t = -.0338$ s

and $t = 6.03$ s

So: The ball is in the air a total of 6.03382 seconds.

e) Find the height at $t = 1.9$ s and at $t = 2.1$ s.

$h(1.9) = 39.171$ m
$h(2.1) = 41.131$ m

$$\text{rate of change} = \frac{(41.131 - 39.171)\text{m}}{(2.1 - 1.9)\text{s}} = 9.8 \text{ m/s}$$

Thus, the velocity at 2 s is 10 m/s.