LAWS OF LOGARITHMS

There are four laws of logarithms that we are going to investigate in this section. These laws are going to enable us to solve equations that we cannot solve as yet.

In looking at these laws, it is a good idea to remember that **logarithms are exponents**. Because of this fact, the basis of these laws are grounded in the rules of exponents.

The proofs of these laws will not be given here.

Law # 1: The Logarithm of a Product

The log of a product to any base is equal to the **sum** of the logs of the factors of that product.

$\log_a(mn) = \log_a m + \log_a n$

↑
Base

Thus: $\log_5 45 = \log_5 9 + \log_5 5$

Also: $\log_5 45 = \log_5 15 + \log_5 3$ and $\log_5 45 = \log_5 90 + \log_5 \frac{1}{2}$

This law applies to any number of factors.

$\log_6 72 = \log_6 2 + \log_6 2 + \log_6 3 + \log_6 3 + \log_6 2$

$\log_a (xyz) = \log_a x + \log_a y + \log_a z$

Law # 2: The Logarithm of a Quotient

The log of a quotient to any base is equal to the **difference** of the logs of the factors of that quotient.

$$\log_a\left(\frac{m}{n}\right) = \log_a m - \log_a n$$

Thus: $\log_3(24) = \log_3 48 - \log_3 2$ since $48/2 = 24$

$\log_3(24) = \log_3 72 - \log_3 3$ since $72/3 = 24$

$\log_3(24) = \log_3 96 - \log_3 2 - \log_3 2$ since $96/2 = 48/2 = 24$

Examples:

1. $\log \frac{xyz}{ab} = \log x + \log y + \log z - \log a - \log b$ The logs of the factors **x, y,** and **z** are **added** since they are in the numerator (multiplied) of the fraction. The logs of the factors **a** and **b** are **subtracted** since they are in the denominator (divided) of the fraction.

2. $\log \frac{5cd}{12xy} = \log 5 + \log c + \log d - \log 12 - \log x - \log y$

Law # 3 **The Logarithm of a Power**

The logarithm of a base raised to any power is equivalent to multiplying the exponent by the logarithm of the base.

$$\log_a b^x = x(\log_a b)$$

Thus: $\log_7 12^5 = 5(\log_7 12)$

$\log_7[(12)(12)(12)(12)(12)]$

$\log_7 12 + \log_7 12 + \ldots$

Law # 4 The Logarithm of a Root

This law is really an extension of Law # 3. Recall from earlier work that expressions with exponents that are rational numbers can be equivalently written as radicals.

$$15^{\frac{2}{3}} = \left(\sqrt[3]{15}\right)^2$$

So, we can write: $\log\left(\sqrt[3]{15}\right)^2 = \log 15^{\frac{2}{3}} = \frac{2}{3}\log 15$ (The base is 10!)

EVALUATING LOGARITHMS USING THE LAWS OF LOGARITHMS

Examples:

Evaluate the following expressions.

1. $\log_5 100 + \log_5 \left(\frac{1}{4}\right)$

   By Law # 1, we can write this as: $\log_5 (100 \cdot \frac{1}{4}) = \log_5 25$

   In the last section we learned how to evaluate $\log_5 25$: $x = \log_5 25$

   $5^x = 25 = 5^2$

   Thus: $\mathbf{x = 2}$

2. $\log_2 384 - \log_2 12$

   By Law # 2, we can write this as: $\log_2 \left(\frac{384}{12}\right) = \log_2 32$

   And: $x = \log_2 32$

   $2^x = 32 = 2^5$ Thus: $\mathbf{x = 5}$

3. $\log_7 49^5$

From Law # 3: $\log_7 49^5 = 5\,(\log_7 49)$ → Now: $\log_7 49 = 2$

Therefore: $= 5\,(2)$

$= \mathbf{10}$

4. $\log_3 \sqrt[5]{27}$

By Law # 4, we can write this as: $\log_3 27^{\frac{1}{5}}$

Which equals: $\frac{1}{5}\log_3 27$

And since $\log_3 27 = 3$: $\frac{1}{5}\cdot 3 = \frac{3}{5}$

Write each expression as a single logarithm.

5. $\log A + \log B - \log C + \log D - \log E$

The plus signs indicate multiplication, and the minus signs indicate division.

$$\log \frac{ABD}{CE}$$

6. $\log A + \log B - (\log C - \log D)$

Simplify first: $\log A + \log B - \log C + \log D$

$$\log \frac{ABD}{C}$$

7. $\log_4 w - 3 \log_4 y + \frac{2}{3} \log_4 z$

$$\log_4 \frac{(wz^{\frac{2}{3}})}{y^3}$$