THE INVERSE OF FUNCTIONS

The ***inverse*** of a function is found by interchanging the x and y values of the function. If the original function is denoted by f(x), the inverse is denoted as $f^{-1}(x)$.

If the function is written as a set of ordered pairs: $f(x) = \{(-1,0), (0,4), (1,7)\}$

Then the inverse is: $f^{-1}(x) = \{(0,-1), (4,0), (7,1)\}$

If the function is written as an equation: $f(x) = y = 7x - 4$

Then the inverse is: $x = 7y - 4$

Which we usually solve for y: $y = \dfrac{x+4}{7} = f^{-1}(x)$

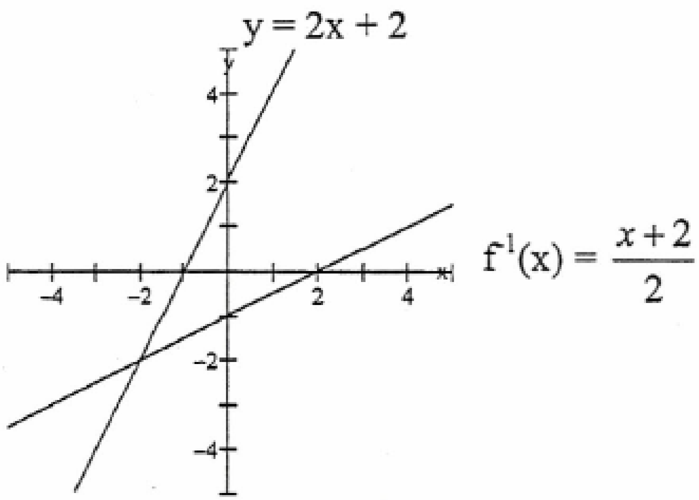
GRAPHS OF INVERSE FUNCTIONS

Inverses of functions are related to their original functions graphically. In fact, we can determine if two functions are inverses of each other simply by looking at their graphs.

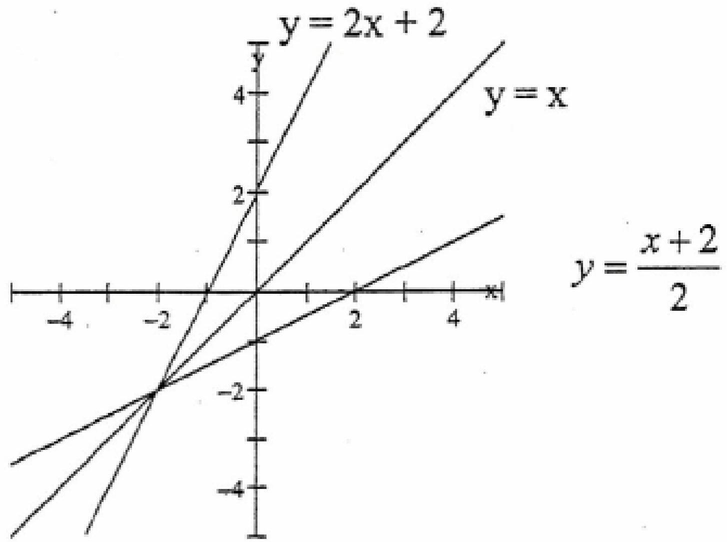
Example 1

Consider the function $y = 2x + 2$. Its inverse would be found by interchanging the x and y values: $x = 2y + 2$, and solve for y: $y = \frac{x-2}{2} = f^{-1}(x)$.

If we graph these two functions, we get the following:



Now we will look at the same two graphs but also include the line y = x. This is a straight line that passes through the origin at a 45° angle.



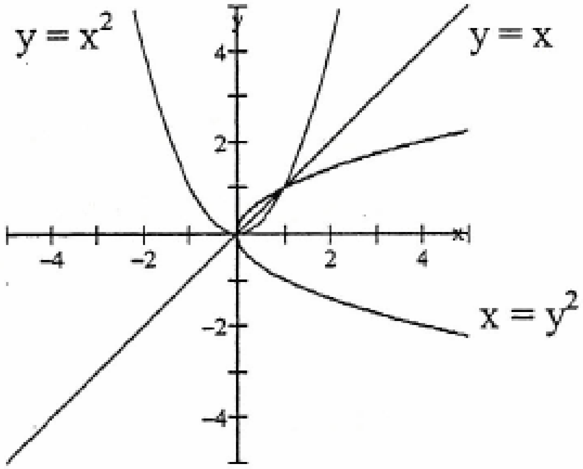
Notice that the inverse function is a mirror of the original function across the line y = x.

We can say that all functions and their inverses are symmetrical about the line y=x.

Example 2:

For the function $y = x^2$, the inverse will be $x = y^2$, or $y = \pm\sqrt{x} = f^{-1}(x)$.

Here are the graphs, as well as the graph of $y = x$.



As you can readily see the two equations are mirrored across the line $y = x$. Thus, they are inverses of each other.

INVERSES OF EXPONENTIAL FUNCTIONS

For the exponential function $y = b^x$, the inverse is $x = b^y$. As we have found in earlier sections, this type of equation can be solved for *y* in certain situations. The method we used was to write both sides of the equations as powers of the same base.

Example 1

Solve $16 = 4^y$

$4^2 = 4^y$

Thus: $y = 2$

But what if we can't write each side as powers of the same base? If we had to solve

$$13 = 4^y,$$

we do not have any algebraic operations to solve for *y*. We could take a calculator and substitute values for *y* until 4^y got closer and closer to 13, but there is a better way.

An operation was created (a long time ago) that enables us to solve for *y* easily. It uses the idea of the **inverse of an exponential function**. The name of this inverse is called the ***logarithmic function***.

Thus, the **exponential function** is: $y = b^x$

The **corresponding logarithmic function** is: $x = b^y$

Which can also be written as: $y = \log_x b$ **(read as "y is the power that x is raised to get b)**

Example 2

We know that, in exponential form, $8 = 2^3$. This can be written in logarithmic form as $3 = \log_2 8$. **The logarithm is the exponent. The base in the exponential form (2) is the base of the logarithm.**

Example 3

Exponential Form	**Logarithmic Form**
$64 = 2^6$	$6 = \log_2 64$
$125 = 5^3$	$3 = \log_5 125$
$\frac{1}{36} = 6^{-2}$	$-2 = \log_6 \left(\frac{1}{36}\right)$
$y = \left(\frac{2}{3}\right)^{-2x}$	$-2x = \log_{\frac{2}{3}} y$
$2m = 45^3$	$3 = \log_{45} (2m)$

Example 4

Evaluate each expression.

a) $\log_5 125$ We wish to find x such that $x = \log_5 125$.

In exponential form: $5^x = 125$

Solving by common bases: $5^x = 5^3$

$x = 3$.

b) $\log_3 27$

$x = \log_3 27$

$3^x = 27 = 3^3$

$x = 3$

c) $\log_{21} 21$

$x = \log_{21} 21$

$21^x = 21$

$x = 1$

Thus: $\log_x x = 1$

d) $\log_7\left(\frac{1}{49}\right)$

$x = \log_7\left(\frac{1}{49}\right)$

$7^x = \left(\frac{1}{49}\right) = 7^{-2}$

$x = -2$

e) $\log_{\frac{1}{4}}(64)$

$x = \log_{\frac{1}{4}}(64)$

$\left(\frac{1}{4}\right)^x = 64$

$2^{-2x} = 2^6$

$-2x = 6$

$x = -3$

SOLVING SIMPLE LOGARITHMIC EQUATIONS

Solve for x

f) $\log_2 x = 5$

$2^5 = x$

$x = 32$

g) $-2 = \log_4 x$

$4^{-2} = x$

$x = \frac{1}{16}$

h) $\log_x 32 = \frac{5}{3}$

$x^{\frac{5}{3}} = 32$

$\left[x^{\frac{5}{3}}\right]^{\frac{3}{5}} = 32^{\frac{3}{5}}$

When the unknown is in the base, we solve by raising both of the equation by the reciprocal of the exponent. This result in the left-hand side reducing to x.

$x = \left(\sqrt[5]{32}\right)^3 = 2^3 = 8$

i) $\log_x 125 = \frac{3}{4}$

$\left[x^{\frac{3}{4}}\right]^{\frac{4}{3}} = 125^{\frac{4}{3}}$

$x = \left(\sqrt[3]{125}\right)^4$

$x = 5^4 = 625$

j) $\log_x \left(\frac{64}{27}\right) = \frac{3}{2}$

$\left[x^{\frac{3}{2}}\right]^{\frac{2}{3}} = \left[\frac{64}{27}\right]^{\frac{2}{3}}$

$x = \left(\sqrt[3]{\frac{64}{27}}\right)^2$

$x = \left(\frac{4}{3}\right)^2 = \frac{16}{9}$

k) $\log_x \left(\frac{1}{36}\right) = -2$

$\left(x^{-2}\right)^{\frac{-1}{2}} = \left(\frac{1}{36}\right)^{\frac{-1}{2}}$

$x = 36^{\frac{1}{2}} = 6$

GRAPHS OF LOGARITHMIC FUNCTIONS

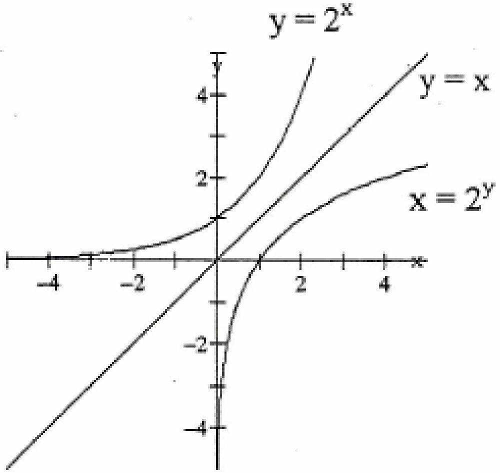
Since logarithmic functions are inverses of exponential functions, we will expect that, when graphed, they will be mirror images of each other across the line $y = x$.

Consider the exponential equation $y = 2^x$.

The inverse will be: $\mathbf{x = 2^y}$ or, as we have found, another way to write this is:

$\mathbf{\log_2 x = y}$

If we graph these two equations, along with the line $y = x$, we obtain the following:



As you can readily see the two equations are mirrored across the line $y = x$. Thus, they are inverses of each other.