Questions from Homework

Develop the definition of a derivative

The concept of **Derivative** is at the core of Calculus and modern mathematics. The definition of the derivative can be approached in two different ways. One is geometrical (as a slope of a curve) and the other one is physical (as a rate of change). Historically there was (and maybe still is) a fight between mathematicians which of the two illustrates the concept of the derivative best and which one is more useful. We will not dwell on this. Our emphasis will be on the use of the derivative as a tool.

Definition. Let $y = f(x)$ be a function. The **derivative of f** is the function whose value at x is the limit

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

provided this limit exists.

If this limit exists for each x in an open interval I, then we say that **f is differentiable on I**.

Derivative → The slope of the tangent

→ $f'(x)$

→ Differentiate

→ y'

→ $\frac{dy}{dx}$

Notation:

$$f'(x) \Leftrightarrow \frac{dy}{dx} \rightarrow \text{First derivative}$$

$$f''(x) \Leftrightarrow \frac{d^2y}{dx^2} \rightarrow \text{Second derivative}$$

Examples:

Use the definition of a derivative to differentiate...

$$\boxed{f'(x) = \lim_{h \to 0} \frac{\boxed{f(x+h)} - \boxed{f(x)}}{h}}$$

$$f(x) = \boxed{4x^2 - 4}$$

$$F(x+h) = 4(x+h)^2 - 4$$

$$F(x+h) = 4(x^2 + 2xh + h^2) - 4$$

$$F(x+h) = \boxed{4x^2 + 8xh + 4h^2 - 4}$$

$$F'(x) = \lim_{h \to 0} \frac{\boxed{F(x+h)} - \boxed{F(x)}}{h}$$

$$= \lim_{h \to 0} \frac{4x^2 + 8xh + 4h^2 - 4 - (4x^2 - 4)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{4x^2} + 8xh + 4h^2 \cancel{-4} \cancel{-4x^2} \cancel{+4}}{h}$$

$$= \lim_{h \to 0} \frac{8xh + 4h^2}{h} \leftarrow \text{Factor}$$

$$F'(x) = \lim_{h \to 0} \frac{\cancel{h}(8x + 4h)}{\cancel{h}} = 8x + 4(0) = \boxed{8x}$$

Slope of tangent

Examples:

Use the definition of a derivative to differentiate...

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

$$f(x) = \boxed{2x^2 - 6x + 1}$$

$$f(x+h) = 2(x+h)^2 - 6(x+h) + 1$$

$$f(x+h) = 2(x^2 + 2xh + h^2) - 6x - 6h + 1$$

$$f(x+h) = \boxed{2x^2 + 4xh + 2h^2 - 6x - 6h + 1}$$

$$f'(x) = \lim_{h \to 0} \frac{2x^2 + 4xh + 2h^2 - 6x - 6h + 1 - (2x^2 - 6x + 1)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{2x^2} + 4xh + 2h^2 \cancel{-6x} - 6h \cancel{+1} \cancel{-2x^2} \cancel{+6x} \cancel{-1}}{h}$$

$$= \lim_{h \to 0} \frac{4xh + 2h^2 - 6h}{h} \quad \leftarrow \text{Factor}$$

$$f'(x) = \lim_{h \to 0} \frac{\cancel{h}(4x + 2h - 6)}{\cancel{h}} = \boxed{4x - 6}$$

Slope of the tangent

Examples:

Use the definition of a derivative to differentiate...

$$\boxed{f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}}$$

$$f(x)=\boxed{\frac{x+1}{3x-2}}$$

$$F(x+h)=\frac{(x+h)+1}{3(x+h)-2}$$

$$F(x+h)=\boxed{\frac{x+h+1}{3x+3h-2}}$$

$$f'(x)=\lim_{h\to 0}\frac{(3x-2)\cancel{(3x+3h-2)}\frac{x+h+1}{\cancel{3x+3h-2}}-\frac{x+1}{\cancel{3x-2}}\cancel{(3x-2)}(3x+3h-2)}{h(3x-2)(3x+3h-2)}$$

$$=\lim_{h\to 0}\frac{(3x-2)(x+h+1)-(x+1)(3x+3h-2)}{h(3x-2)(3x+3h-2)}$$ Expand

$$=\lim_{h\to 0}\frac{3x^2+3xh+3x-2x-2h-2-(3x^2+3xh-2x+3x+3h-2)}{h(3x-2)(3x+3h-2)}$$

$$=\lim_{h\to 0}\frac{\cancel{3x^2}\cancel{+3xh}\cancel{+3x}\cancel{-2x}-2h\cancel{-2}\cancel{-3x^2}\cancel{-3xh}\cancel{+2x}\cancel{-3x}-3h\cancel{+2}}{h(3x-2)(3x+3h-2)}$$

$$=\lim_{h\to 0}\frac{-5\cancel{h}}{\cancel{h}(3x-2)(3x+3h-2)}=\frac{-5}{(3x-2)(3x-2)}$$

$$\boxed{=\frac{-5}{(3x-2)^2}}$$

Find the derivative of each function.

Remember!

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

1. $f(x) = 8x^2 - 10$

2. $f(x) = 2x^2 + 14x - 7$

3. $f(x) = x^3$

4. $f(x) = \dfrac{x+4}{2x+3}$