Pascal's Triangle and the Expansion of Binomials

An important application of Pascal's Triangle is in the area of binomial expansion. Consider the following expansion of binomials:

$(x + 1)^2 = (x + 1)(x + 1) = x^2 + 2x + 1$

$(x + 1)^3 = (x + 1)(x + 1)(x + 1) = x^3 + 3x^2 + 3x + 1$

$(x + 1)^4 = (x + 1)(x + 1)(x + 1)(x + 1) = x^4 + 4x^3 + 6x^2 + 4x + 1$

As the exponent increases, the math becomes more cumbersome. Pascal's Triangle gives us a way to expand binomials raised to any exponent.

If we look at the coefficients of the expansions in the above examples:

For $(x + 1)^2$ the coefficients are: 1 2 1

For $(x + 1)^3$ the coefficients are: 1 3 3 1

For $(x + 1)^4$ the coefficients are: 1 4 6 4 1

These coefficients are rows 2, 3, and 4 of Pascal's Triangle! Remember that these can be found using Combinations!!!

The coefficients are then multiplied by the terms x^4, x^3, x^2, and x for these examples. These will change according to the binomial being expanded. To expand binomials in this fashion, we are using the Binomial Theorem.

The Binomial Theorem

The expansion of $(a + b)^n$, where *n* is a natural number, is given by:

$$(a + b)^n = {}_nC_0\, a^n\, b^0 + {}_nC_1\, a^{n-1}\, b^1 + {}_nC_2\, a^{n-2}\, b^2 + \dots + {}_{n-1}C_1\, a^1\, b^{n-1} + {}_nC_n\, a^0\, b^n$$

To illustrate this more clearly, let's look at a few examples:

Example 1

Expand $(x + 2)^4$ → 4th row

Solution

$a = x$
$b = 2$

Since we are dealing with the 4th row of Pascal's Triangle the coefficients would be:

$_4C_0$	$_4C_1$	$_4C_2$	$_4C_3$	$_4C_4$
1	**4**	**6**	**4**	**1**

The coefficients need to be multiplied by certain terms to complete the expansion:

The first coefficient will be multiplied by $(x)^4(2)^0$

The second coefficient will be multiplied by $(x)^3(2)^1$

The third coefficient will be multiplied by $(x)^2(2)^2$

The fourth coefficient will be multiplied by $(x)^1(2)^3$

The fifth coefficient will be multiplied by $(x)^0(2)^4$

Thus, the expansion of $(x + 2)^4$ is:

$(1)(x)^4(2)^0 + (4)(x)^3(2)^1 + (6)(x)^2(2)^2 + (4)(x)^1(2)^3 + (1)(x)^0(2)^4$

$= (1)(x^4)(1) + (4)(x^3)(2) + (6)(x^2)(4) + (4)(x^1)(8) + (1)(1)(16)$

$\mathbf{= 1x^4 + 8x^3 + 24x^2 + 32x^1 + 16}$

Example 2

Expand $(x + 3)^5$

Solution

$${}_5C_0(x)^5(3)^0 + {}_5C_1(x)^4(3)^1 + {}_5C_2(x)^3(3)^2 + {}_5C_3(x)^2(3)^3 + {}_5C_4(x)^1(3)^4 + {}_5C_5(x)^0(3)^5$$

$$= (1)(x^5)(1) + (5)(x^4)(3) + (10)(x^3)(9) + (10)(x^2)(27) + (5)(x^1)(81) + (1)(1)(243)$$

$$\mathbf{= 1x^5 + 15x^4 + 90x^3 + 270x^2 + 405x^1 + 243}$$

$(x+2)^5$ Row 5

$a = x$

$b = 2$

$${}_5C_0(x^5)(2^0) + {}_5C_1(x^4)(2^1) + {}_5C_2(x^3)(2^2) + {}_5C_3(x^2)(2^3) + {}_5C_4(x^1)(2^4) + {}_5C_5(x^0)(2^5)$$

$$1(x^5)(1) + 5(x^4)(2) + 10(x^3)(4) + 10(x^2)(8) + 5(x^1)(16) + (1)(1)(32)$$

$$\boxed{x^5 + 10x^4 + 40x^3 + 80x^2 + 80x + 32}$$