Problem of the Week
Grade 11 and 12

Five Prime Mates
Solution

Problem

The product of five different **odd** prime numbers is a five-digit number of the form $strst$, where $r = 0$. Determine all possible numbers.

Solution 1

Be sure to read Solution 2 after Solution 1

A number is divisible by 11 if the difference between the sum of the even position numbers and sum of the odd position numbers is a multiple of 11. (This problem can still be done without knowing this divisibility fact but the task is made simpler with it.) The sum of the digits in the even positions is $strst$ is $t + s$. The sum of the digits in the odd positions is $s + r + t$ but $r = 0$ so the sum is $s + t$. The difference of the two sums is $(t + s) - (s + t) = 0$ which is a multiple of 11. Therefore, $st0st$ is divisible by 11.

Only odd factors are used so the product will be odd. This means that the product looks like _10_1, _30_3, _50_5, _70_7, or _90_9. So we begin to systematically look at the possibilities.

First, we will examine numbers that have 3 (and 11) as a factor. To be divisible by three, the sum of the digits will be divisible by three. To be divisible by nine, the sum of the digits will be divisible by nine. But if the number is divisible by nine, it is divisible by 3^2 and would have a repeated prime factor which is not allowed. So we want numbers divisible by 3 but not 9. The possibilities are as follows: 21021, 51051, 33033, 93093, 15015, 75075, 57057, 87087, 39039, and 69069. The sum of the digits of these numbers is divisible by three so the numbers are divisible by three. The numbers 81081, 63063, 45045, 27027 and 99099 are divisible by 9 and have therefore been eliminated.

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$21021 = 3 \times 11 \times 637 = 3 \times 11 \times 7 \times 91 = 3 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$51051 = 3 \times 11 \times 1547 = 3 \times 11 \times 7 \times 221 = 3 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 51051 is a valid number.

$$33033 = 3 \times 11 \times 1001 = 3 \times 11 \times 7 \times 143 = 3 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$93093 = 3 \times 11 \times 2821 = 3 \times 11 \times 7 \times 403 = 3 \times 11 \times 7 \times 13 \times 31$$

Since there are 5 different odd prime factors, 93093 is a valid number.

$$15015 = 3 \times 11 \times 455 = 3 \times 11 \times 5 \times 91 = 3 \times 11 \times 5 \times 7 \times 13$$

Since there are 5 different odd prime factors, 15015 is a valid number.

$$75075 = 3 \times 11 \times 2275 = 3 \times 11 \times 5 \times 455 = 3 \times 11 \times 5 \times 5 \times 91 = 3 \times 11 \times 5 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated and there are six prime factors, this is not a valid number.

$$57057 = 3 \times 11 \times 1729 = 3 \times 11 \times 7 \times 247 = 3 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 57057 is a valid number.

$$87087 = 3 \times 11 \times 2639 = 3 \times 11 \times 7 \times 377 = 3 \times 11 \times 7 \times 13 \times 29$$

Since there are 5 different odd prime factors, 87087 is a valid number.

$$39039 = 3 \times 11 \times 1183 = 3 \times 11 \times 7 \times 169 = 3 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$69069 = 3 \times 11 \times 2093 = 3 \times 11 \times 7 \times 299 = 3 \times 11 \times 7 \times 13 \times 23$$

Since there are 5 different odd prime factors, 69069 is a valid number.

Second, we will examine numbers that are divisible by 5 but not 3, since divisibility by three has been examined. If a number is divisible by 5 it ends in 5 or 0. Since the number is odd, we can exclude any number ending in 0 leaving 25025, 35035, 55055, 65065, 85085 and 95095 as possible numbers. (15015, 45045, 75075 were examined above and have been excluded.)

Now we examine the prime factorization of each of these numbers to see which numbers satisfy the conditions.

$$25025 = 5 \times 11 \times 455 = 5 \times 11 \times 5 \times 91 = 5 \times 11 \times 5 \times 7 \times 13$$

Since the prime factor 5 is repeated, this is not a valid number.

$$35035 = 5 \times 11 \times 637 = 5 \times 11 \times 7 \times 91 = 5 \times 11 \times 7 \times 7 \times 13$$

Since the prime factor 7 is repeated, this is not a valid number.

$$55055 = 5 \times 11 \times 1001 = 5 \times 11 \times 7 \times 143 = 5 \times 11 \times 7 \times 11 \times 13$$

Since the prime factor 11 is repeated, this is not a valid number.

$$65065 = 5 \times 11 \times 1183 = 5 \times 11 \times 7 \times 169 = 5 \times 11 \times 7 \times 13 \times 13$$

Since the prime factor 13 is repeated, this is not a valid number.

$$85085 = 5 \times 11 \times 1547 = 5 \times 11 \times 7 \times 221 = 5 \times 11 \times 7 \times 13 \times 17$$

Since there are 5 different odd prime factors, 85085 is a valid number.

$$95095 = 5 \times 11 \times 1729 = 5 \times 11 \times 7 \times 247 = 5 \times 11 \times 7 \times 13 \times 19$$

Since there are 5 different odd prime factors, 95095 is a valid number.

Thirdly, we will look at numbers that are divisible by 7 but not 3 or 5. If we multiply 7 by the next four odd prime numbers we get $7 \times 11 \times 13 \times 17 \times 19 = 323323$, a six digit number so we are beyond all possible solutions.

Therefore there are 8 numbers of the form $st0st$ which are the product of five different odd prime numbers, namely 51051, 93093, 15015, 57057, 87087, 69069, 85085 and 95095.

Look at Solution 2 for a much more insightful approach to the problem.

Problem

The product of five different **odd** prime numbers is a five-digit number of the form $strst$, where $r = 0$. Determine the number of possible numbers.

Solution 2

Since $r = 0$, the number is of the form $st0st$. Then

$$st0st = st(1000) + st = st(1000 + 1) = st(1001)$$

This means that the number $st0st$ is divisible by 1001 which is the product of the three odd prime factors 7, 11, and 13. So st is a two digit number which is the product of two different odd prime factors none of which can be 7, 11 or 13. It is now a straight forward matter of generating all possible two digit products, $a \times b$ say, using odd prime factors other than 7, 11 and 13.

Prime Factor a	Prime Factor b	st $= a \times b$	Five Different Odd Primes	Product $st0st$
3	5	15	$3, 5, 7, 11, 13$	15015
3	17	51	$3, 7, 11, 13, 17$	51051
3	19	57	$3, 7, 11, 13, 19$	57057
3	23	69	$3, 7, 11, 13, 23$	69069
3	29	87	$3, 7, 11, 13, 29$	87087
3	31	93	$3, 7, 11, 13, 31$	93093
5	17	85	$5, 7, 11, 13, 17$	85085
5	19	95	$5, 7, 11, 13, 19$	95095

No other two digit product of two different odd prime factors other than 7, 11 and 13 exists.

Therefore there are 8 numbers of the form $st0st$ which are the product of five different odd prime numbers, namely 15015, 51051, 57057, 69069, 87087, 93093, 85085 and 95095.

Look on the following page for a summary of the divisibility tests for the numbers 2 through 12.

Divisibility Tests

Divisibility by 2: A number is divisible by 2 if the last digit is even.

Divisibility by 3: A number is divisible by 3 if the sum of the digits is divisible by 3. For example, 1295 is not divisible by 3 since $1 + 2 + 9 + 5 = 17$ which is not divisible by 3. However, 1296 is divisible by 3 since $1 + 2 + 9 + 6 = 18$ which is divisible by 3.

Divisibility by 4: A number is divisible by 4 if the last two digits are divisible by 4. For example, 1295 is not divisible by 4 since 95 is not divisible by 4. However, 1296 is divisible by 4 since 96 is divisible by 4.

Divisibility by 5: A number is divisible by 5 if the last digit is a 0 or 5.

Divisibility by 6: A number is divisible by 6 if it is divisible by both 2 and 3. The number 395 is not divisible by 6 since it is not even and hence is not divisible by 2. The number 862 is not divisible by 6 since it is not divisible by 3 ($8 + 6 + 2 = 16$ which is not divisible by 3). The number 2964 is divisible by 6. It is even and is therefore divisible by 2. It is divisible by 3 since $2 + 9 + 6 + 4 = 21$ which is divisible by 3. Since 2964 is divisible by both 2 and 3, it is divisible by 6.

Divisibility by 7: We can follow an unusual algorithm to determine if an number is divisible by 7: Remove the unit's digit, double that digit and subtract it from the leftover number. If the difference is divisible by 7, the original number is divisible by seven. If unsure, repeat the algorithm with the new number.

Is 1356 divisible by 7? Remove the 6, double the 6 to 12, subtract from 135 leaving 123. Is 123 divisible by 7? Remove the 3, double the 3 to 6, subtract from 12 leaving 6. 6 is not divisible by 7 and therefore 1356 is not divisible by 7.

Is 45 024 divisible by 7? Remove the 4, double to 8, subtract from 4502 giving 4494. Repeat. Remove the 4, double to 8, subtract from 449 giving 441. Repeat. Remove the 1, double to 2, subtract from 44 giving 42 which is divisible by 7. Therefore, 45 024 is divisible by 7.

Divisibility by 8: A number is divisible by 8 if the last three digits are divisible by 8. For example, 1295 is not divisible by 8 since 295 is not divisible by 8. However, 1296 is divisible by 8 since 296 is divisible by 8.

Divisibility by 9: A number is divisible by 9 if the sum of the digits is divisible by 9. For example, 1295 is not divisible by 9 since $1 + 2 + 9 + 5 = 17$ which is not divisible by 9. However, 1296 is divisible by 9 since $1 + 2 + 9 + 6 = 18$ which is divisible by 9.

Divisibility by 10: A number is divisible by 10 if the last digit is a 0.

Divisibility by 11: We can follow an unusual algorithm to determine if an number is divisible by 11: Add the numbers in the even positions. Add the numbers in the odd positions. Subtract the two sums. If this difference is divisible by 11, the original number is divisible by 11.

Is 1 235 862 divisible by 11? The sums are $1 + 3 + 8 + 2 = 14$ and $2 + 5 + 6 = 13$. The difference of the sums is 1, which is not divisible by 11. Therefore, the number 1 235 862 is not divisible by 11.

Is 4 151 617 151 divisible by 11? The sums are $1 + 1 + 1 + 1 + 1 = 5$ and $4 + 5 + 6 + 7 + 5 = 27$. The difference of the sums is -22, which is divisible by 11. Therefore, the number 4 151 617 151 is divisible by 11.

Is 7 326 495 divisible by 11? The sums are $7 + 2 + 4 + 5 = 18$ and $3 + 6 + 9 = 18$. The difference of the sums is 0, which is divisible by 11. Therefore, the number 7 326 495 is divisible by 11.

Divisibility by 12: A number is divisible by 12 if it is divisible by 4 and 3. The number 394 is not divisible by 12 since 94 is not divisible by 4. The number 964 is not divisible by 12 since it is not divisible by 3. (The sum of the digits is 19 which is not divisible by 3.) The number 2964 is divisible by 12. The last two digits, 64, are divisible by 4 and therefore 2964 is divisible by 4. The sum of the digits is 21 which is divisible by 3. Since 2964 is divisible by 4 and 3, it is divisible by 12.