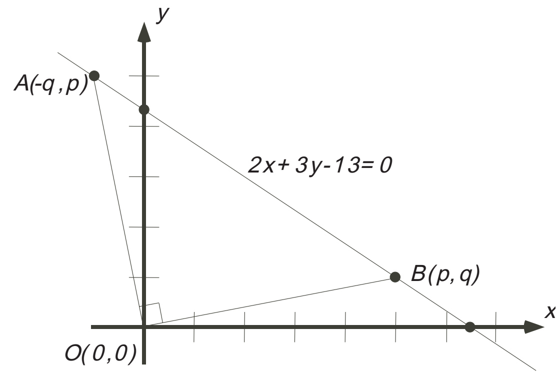
Problem of the Week
Grade 11 and 12

Get Your Analytic Skills On
Solution



Problem

OAB is an isosceles right triangle with vertex O at the origin $(0,0)$, vertices A and B on the line $2x+3y-13=0$ and $\angle AOB = 90^\circ$. Determine the area of $\triangle OAB$.

Solution

In analytic geometry problems, a representative diagram is important and often provides clues for the solution of the problem. The diagram above has the given information plus a couple of pieces of information that will be justified now.

Let B have coordinates (p,q). The slope of $OB = \frac{q}{p}$. Since $\angle AOB = 90^\circ$, $OB \perp OA$ and the slope of OA is the negative reciprocal of OB. Therefore the slope of $OA = \frac{p}{-q}$. Since the triangle is isosceles, $OA = OB$ and it follows that the coordinates of A are $(-q,p)$. (We can verify this by finding the length of OA and the length of OB and showing that both are equal to $\sqrt{p^2+q^2}$.)

Since $B(p,q)$ is on the line $2x+3y-13=0$, it satisfies the equation of the line.

$$\therefore 2p+3q-13=0 \quad (1)$$

Since $A(-q,p)$ is on the line $2x+3y-13=0$, it satisfies the equation of the line.

$$\therefore -2q+3p-13=0 \text{ or } 3p-2q-13=0 \quad (2)$$

Since we have two equations and two unknowns, we can use elimination to solve for p and q.

$$\begin{array}{rrcl} (1)\times 2 & 4p+6q-26 & = & 0 \\ (2)\times 3 & 9p-6q-39 & = & 0 \\ \text{Adding, we obtain} & 13p-65 & = & 0 \\ & \therefore p & = & 5 \\ \text{Substituting in } (1) & 10+3q-13 & = & 0 \\ & 3q & = & 3 \\ & \therefore q & = & 1 \end{array}$$

The point B is $(5,1)$ and the length of $OB = \sqrt{5^2+1^2} = \sqrt{26}$. Since $OA = OB$, $OA = \sqrt{26}$. $\triangle AOB$ is a right triangle so we can use OB as the base and OA as the height in the formula for the area of a triangle. Then the area of $\triangle AOB = \frac{OA \times OB}{2} = \frac{\sqrt{26}\sqrt{26}}{2} = 13$.

$\therefore$ the area of $\triangle AOB$ is 13 units^2.