Problem of the Week
Grade 9 and 10

Formidable Fractions?
Solution

Problem

For positive integers a and c, $\dfrac{\left(\frac{a}{c}+\frac{a}{2}+1\right)}{\left(\frac{2}{a}+\frac{2}{c}+1\right)}=18$. Determine the number of ordered pairs (a,c) that satisfy $a+3c\leq 99$.

Solution 1

First we should simplify the fractional equation, $\dfrac{\left(\dfrac{a}{c}+\dfrac{a}{2}+1\right)}{\left(\dfrac{2}{a}+\dfrac{2}{c}+1\right)}=18$

Finding common denominators, $\dfrac{\left(\dfrac{2a}{2c}+\dfrac{ac}{2c}+\dfrac{2c}{2c}\right)}{\left(\dfrac{2c}{ac}+\dfrac{2a}{ac}+\dfrac{ac}{ac}\right)}=18$

Simplifying, $\dfrac{\left(\dfrac{2a+ac+2c}{2c}\right)}{\left(\dfrac{2c+2a+ac}{ac}\right)}=18$

Multiplying by the reciprocal, $\dfrac{(2a+ac+2c)}{2c}\times\dfrac{ac}{(2c+2a+ac)}=18$

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to $\dfrac{ac}{2c}=18$. Since $c\neq 0$, the expression further simplifies to $\dfrac{a}{2}=18$ or $a=18(2)=36$. Substituting $a=36$ into $a+3c\leq 99$ we obtain $36+3c\leq 99$ which simplifies to $3c\leq 63$ and $c\leq 21$ follows.

But $c\geq 1$ and c is an integer so $1\leq c\leq 21$. The value of a is 36 for each of the 21 possible values of c.

$\therefore$ there are 21 ordered pairs (a,c) that satisfy the problem.

Solution 2 is on the next page.

Solution 2

First we should simplify the fractional equation, $\dfrac{\left(\dfrac{a}{c}+\dfrac{a}{2}+1\right)}{\left(\dfrac{2}{a}+\dfrac{2}{c}+1\right)}=18$

Multiply the numerator and the denominator by $2ac$: $\dfrac{\left(\dfrac{a}{c}+\dfrac{a}{2}+1\right)}{\left(\dfrac{2}{a}+\dfrac{2}{c}+1\right)}\times\dfrac{2ac}{2ac}=18$

The equation simplifies to: $\dfrac{2a^2+a^2c+2ac}{4c+4a+2ac}=18$

Factoring, we obtain: $\dfrac{a(2a+ac+2c)}{2(2c+2a+ac)}=18$

Since the bracketed numerator and bracketed denominator are the same and cannot equal zero, we can simplify to $\dfrac{a}{2}=18$ and $a=36$ follows.

Substituting $a=36$ into $a+3c\leq 99$ we obtain $36+3c\leq 99$ which simplifies to $3c\leq 63$ and $c\leq 21$. But $c\geq 1$ and c is an integer so $1\leq c\leq 21$. The value of a is 36 for each of the 21 possible values of c.

$\therefore$ there are 21 ordered pairs (a,c) that satisfy the problem.