EXPONENTIAL FUNCTIONS

The main focus of this unit is to examine the characteristics of **exponential functions**. The simplest form of exponential functions can be described using the function $y = b^x$, where b is the **base** and x is the **exponent** or **power**. Furthermore, $b > 0$.

We are going to begin our examination of simple exponential functions by looking at the tables of values for a few functions. We will try to relate these functions to the geometric sequences introduced in the last section.

Example 1

Give the table of values for the functions: $\boxed{y = 2^x}$, $y = 5^x$, and $y = 0.5^x$. For the domain, use the successive whole number values x = {0, 1, 2, 3, 4, 5}.

x	2^x	5^x	0.5^x
0	1	1	1
1	2	5	0.5
2	4	25	0.25
3	8	125	0.125
4	16	625	0.0625
5	32	3125	0.03125

Consider the y-values for $y = 2^x$. Take any term and divide it by the term before it. The value of this quotient is always 2.

$$\frac{2}{1} = \frac{4}{2} = \frac{8}{4} = \frac{16}{8} = \frac{32}{16} = 2$$

For $y = 5^x$, $\quad \frac{5}{1} = \frac{25}{5} = \frac{125}{25} = \frac{625}{125} = \frac{3125}{625} = 5$

and for $y = 0.5^x$ $\quad \frac{.5}{1} = \frac{.25}{.5} = \frac{.125}{.25} = \frac{.0625}{.125} = \frac{.03125}{.0625} = 0.5$

We can say from these examples that for exponential functions the quotient of one term to the one before it is a **constant.**

Relating this to the previous section, we can see that the y-values of exponential functions are **geometric sequences.** The constant values that we found for the three exponential functions above (2, 5, 0.5) are the **common ratios** that we discussed earlier. are equal to base

To check if a function is exponential then, we divide one y-value by the one before it, and if the quotient is a constant throughout, then the function is exponential. This is assuming, of course, that the x-values change by equal increments.

Example 2

Is this set of ordered pairs exponential? {(-1, 4), (0, 16), (1, 64), …}. If so, what is the ratio of successive y-values?

x	-1	0	1	
y	4	16	64	

4 4

Solution

Yes, since the x-values are successive, and $\frac{16}{4} = \frac{64}{16} = 4$. The common ratio is 4.

$r = 4$

Example 3

Is this set of ordered pairs exponential? {(-1, 4), (1, 16), (3, 64), …}. If so, what is the ratio of successive y-values?

x	-1	1	3	
y	4	16	64	

4 4

Solution

The x-values change by equal increments, and $\frac{16}{4} = \frac{64}{16} = 4$. So, this is an exponential function. **However, the x-values are not successive.** Thus, the common ratio is not 4, since the common ratio is found by dividing successive values.

$r = \sqrt{4} = 2$

If we wrote the ordered pairs in successive order, we can see that there is one ordered pair between the given pairs (-1, 4) and (1, 16). That is the ordered pair that has an x-value of 0.

Thus, {(-1, 4), (0, y), (1, 16),...}. The first y-value is multiplied by 4 to get the third y-value. It makes sense, then, to get the 2nd y-value, the first value has to be multiplied by 2, which is the **square root** of 4.

Thus, the ratio of successive y-values for this function is 2.

In general,

- **If the x-values increase by 1, the common ratio of successive y-values of the function can be found by dividing any term by the one before it.**
- **If the x-values increase by 2, the common ratio of successive y-values of the function can be found by dividing any term by the one before it, *and* taking the square root of that value.**
- **If the x-values increase by 3, the common ratio of successive y-values of the function can be found by dividing any term by the one before it, *and* taking the cube root of that value.**

This can be extended to include x-values increased by any increment.

Example 4

a) Given

x	1	2	3	4
y	5	20	80	320

4 4 4

$r = 4$

The x-values increase by 1, so the common ratio is $\frac{20}{5} = 4$.

b) Given

x	1	3	5	7
y	1	7	49	343

$r = \sqrt{7}$

7 7 7

The x-values increase by 2, so the common ratio is $\sqrt{\frac{49}{7}} = \sqrt{7}$.

c) Given

x	1	5	9	13
y	1	$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$

$r = \sqrt[4]{1/2}$

1/2 1/2 1/2

The x-values increase by 4, so the common ratio is $\sqrt[4]{\frac{\frac{1}{8}}{\frac{1}{4}}} = \sqrt[4]{\frac{1}{2}}$.

Example 5

Sarah borrows \$1200 from a friend, who charges her 10% interest compounded yearly.

a) Make a table of values showing the amount of money she owes at the end of each year for 6 years, assuming she does not pay any money to her friend over this time. Your first table entry should be for year 0, when she owes \$1200.

b) Find how much she would owe in 15 years, if she did not pay any money back.

Solution

a)

x	0	1	2	3	4	5	6
y	\$1200	\$1320	\$1452	\$1597.20	\$1756.92	\$1932.62	\$2125.87

1.1 1.1 1.1 1.1 1.1 1.1

$\boxed{r = 1.1}$

Notice that the **x-values change by successive increments** (1) and **the quotient of one term to the one before it is the constant 1.1.** Thus, this represents a **geometric sequence** where $r = 1.1$.

b) We could continue the above table, multiplying each term by 1.1 until we obtained the amount owing when x = 15. The better way to do it would be to find the equation of the exponential function that represents this set of data so we could use it to find out the amount owed at the end of 15 years, or 20 years, etc.

Look at the y-values of the table. Notice that:

For x = 0,	$1200 = 1200 \times 1.1^0$
For x = 1	$1320 = 1200 \times 1.1^1$
For x = 2	$1452 = 1200 \times 1.1^2$
For x = 3	$1597.20 = 1200 \times 1.1^3$
For x = 4	$1756.92 = 1200 \times 1.1^4$
For x = 5	$1932.62 = 1200 \times 1.1^5$
For x = 6	$2125.87 = 1200 \times 1.1^6$

Thus, it seems that the formula for this exponential function is $\mathbf{y = 1200(1.1)^x}$.

So, at the end of 15 years, she will owe $1200(1.1)^{15} = \mathbf{\$5012.70}$.