RATIONAL EXPONENTS

In our discussion of exponential functions and equations thus far, we have only considered exponents which were integers, that is, …, -3, -2, -1, 0, 1, 2, 3, …

We extend our knowledge of exponents in this section by considering exponents that are **rational numbers**. A ***rational number*** is any number that can be written as a ratio of integers.

Some examples of rational numbers are: $\frac{2}{3}, \frac{-5}{13}, \frac{0}{12}, \frac{-12}{-87}, \ldots$

In the definition, it says that a rational number is any number that ***can*** be written as a ratio of integers.

Thus, the following are also rational numbers: $15 = \frac{15}{1}$, $0 = \frac{0}{114}$, $\sqrt{25} = \frac{5}{1}$.

There are some real numbers that are **not rational:** $\sqrt{13}$, $\sqrt{43}$, π

And, of course, some numbers are not real, and therefore, **not rational:** $\frac{7}{0}$, $\sqrt{-25}$

The basic rule we use to evaluate expressions with rational exponents is the following:

$$b^{\frac{m}{n}} = \left(\sqrt[n]{b}\right)^m$$

What this does is change the expression to an equivalent expression in radical form. A ***radical*** is a number in the form $\sqrt{5}, \sqrt[4]{13}, \sqrt[6]{112}$, etc.

A radical, then, is any root of a number. It can be square (2^{nd}) root, cube (3^{rd}) root, 4^{th}-root, etc. **If we want the square root of a number, we do not include the number 2 in our expression.**

For example, $\sqrt{49}$ is taken to mean the square root of 49 = 7.

If we want a different root, we have to indicate such: $\sqrt[3]{8} = 2$ or $\sqrt[5]{32} = 2$.

Examples:

Write the following as radicals.

1. $13^{\frac{2}{5}} = \left(\sqrt[5]{13}\right)^2$

2. $-123^{\frac{9}{2}} = -\left(\sqrt[2]{123}\right)^9$ Recall that the exponent only applies to the base directly below it. $-123^{\frac{9}{2}}$ can be thought of as: $(-1)(123)^{\frac{9}{2}}$.

3. $(-17)^{\frac{1}{5}} = \sqrt[5]{-17}$ The brackets mean that the base is (-17).

4. $\left(\frac{5}{6}\right)^{\frac{4}{5}} = \left(\sqrt[5]{\frac{5}{6}}\right)^4$

5. $\left(\frac{2}{7}\right)^{\frac{-6}{11}} = \left(\frac{7}{2}\right)^{\frac{6}{11}} = \left(\sqrt[11]{\frac{7}{2}}\right)^6$ Recall that a fraction raised to a negative exponent can be written as the reciprocal of the base raised to the positive value of the exponent.

Write each using positive fractional exponents.

6. $\sqrt[5]{14} = 14^{\frac{1}{5}}$

7. $(\sqrt[3]{111})^4 = 111^{\frac{4}{3}}$

8. $\dfrac{1}{\sqrt{13}} = \dfrac{1}{13^{\frac{1}{2}}}$

9. $\dfrac{1}{\sqrt[7]{5^{-x}}} = \dfrac{1}{5^{\frac{-x}{7}}} = 5^{\frac{x}{7}}$

Evaluate without using a calculator.

10. $25^{\frac{1}{2}} = \sqrt{25} = 5$

11. $-36^{\frac{1}{2}} = -\sqrt{36} = -6$

12. $(-36)^{\frac{1}{2}} = \sqrt{-36}$ = no real solution. There is no real solution when you take an even root of a negative quantity. Thus, $\sqrt[4]{-56}, \sqrt[6]{-47}$, have no real solutions.

13. $(-8)^{\frac{-1}{3}} = \sqrt[3]{-8} = -2$, since (-2)(-2)(-2) = -8. There is a real solution when you take an odd root of a negative quantity.

14. $\left(\frac{625}{81}\right)^{\frac{1}{4}} = \sqrt[4]{\frac{625}{81}} = \frac{5}{3}$

15. $\left(\frac{4}{49}\right)^{\frac{-3}{2}} = \left(\frac{49}{4}\right)^{\frac{3}{2}} = \left(\sqrt[2]{\frac{49}{4}}\right)^{3} = \left(\frac{7}{2}\right)^{3} = \frac{343}{8}$

16. $16^{\frac{3}{2}} + 27^{\frac{-1}{3}} - 4^{\frac{5}{2}}$

$\left(\sqrt{16}\right)^{3} + \frac{1}{\sqrt[3]{27}} - \left(\sqrt{4}\right)^{5}$

$4^{3} + \frac{1}{3} - 2^{5}$

$64 + \frac{1}{3} - 32 = 32 + \frac{1}{3} = \frac{97}{3}$

Rational Exponents

$\sqrt{x} = x^{1/2}$

$\sqrt[2]{4} = (4)^{1/2} = 2$

$\sqrt[3]{x} = x^{1/3}$

$\sqrt[4]{x} = x^{1/4}$

$\sqrt[5]{x} = x^{1/5}$

$\sqrt[5]{x^2} = x^{2/5}$

$\sqrt[7]{x^3} = x^{3/7}$

$\sqrt[3]{x^7} = x^{7/3}$

$\sqrt{x^{-5}} = x^{-5/2} = \left(\frac{1}{x}\right)^{5/2} = \frac{1}{x^{5/2}}$

$\sqrt[3]{\left(\frac{2}{3}\right)^{-2}} = \left(\frac{2}{3}\right)^{-2/3} = \left(\frac{3}{2}\right)^{2/3}$

Ex: 6

⑦ $(x^{-1})^{m/n}$

$x^{-m/n}$

$\sqrt[n]{x^{-m}}$

㉘ $27^{2/3} + 16^{3/4} - \left(\frac{1}{3}\right)^{-1} + (-3)^0$

$(3^3)^{2/3} + (2^4)^{3/4} - (3)^1 + 1$

$3^2 + 2^3 - 3 + 1$

$9 + 8 - 3 + 1$

$\boxed{15}$