# Radical Functions and Transformations

## Focus on...

- investigating the function $y = \sqrt{x}$ using a table of values and a graph
- graphing radical functions using transformations
- identifying the domain and range of radical functions

**radical function**

- a function that involves a radical with a variable in the radicand
- $y = \sqrt{3x}$ and $y = 4\sqrt[3]{5 + x}$ are radical functions.

## Example 1

### Graph Radical Functions Using Tables of Values

$y = a\sqrt{b(x-h)} + k$

Use a table of values to sketch the graph of each function. Then, state the domain and range of each function.

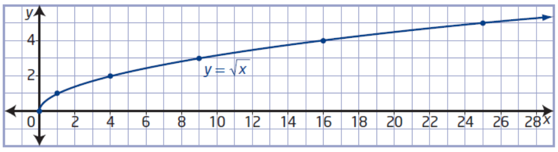
**a)** $y = \sqrt{x}$ **b)** $y = \sqrt{x - 2}$ **c)** $y = \sqrt{x} - 3$

(base function)

**a)** For the function $y = \sqrt{x}$, the radicand $x$ must be greater than or equal to zero, $x \geq 0$. (cannot take the square root of a negative)

| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |
| 25 | 5 |

How can you choose values of $x$ that allow you to complete the table without using a calculator? → use Perfect squares




The graph has an endpoint at (0, 0) and continues up and to the right. The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

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**b)** For the function $y = \sqrt{x - 2}$, the value of the radicand must be greater than or equal to zero.

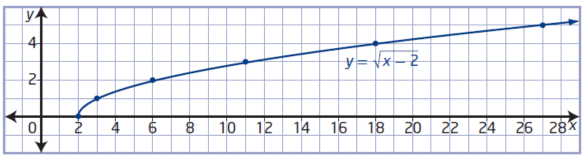
$x - 2 \geq 0$
$x \geq 2$

$h = 2$
Translated 2 units to the right

| x | y |
|---|---|
| 2 | 0 |
| 3 | 1 |
| 6 | 2 |
| 11 | 3 |
| 18 | 4 |
| 27 | 5 |

How is this table related to the table for $y = \sqrt{x}$ in part a)?

How does the graph of $y = \sqrt{x - 2}$ compare to the graph of $y = \sqrt{x}$?




The domain is $\{x \mid x \geq 2, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

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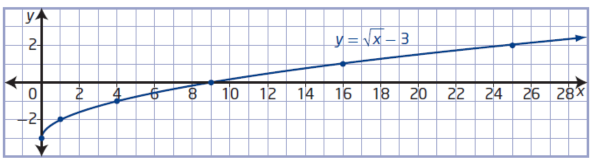
**c)** The radicand of $y = \sqrt{x} - 3$ must be non-negative.
$x \geq 0$

$k = -3$
Translated 3 units down

| x | y |
|---|---|
| 0 | −3 |
| 1 | −2 |
| 4 | −1 |
| 9 | 0 |
| 16 | 1 |
| 25 | 2 |

How does the graph of $y = \sqrt{x} - 3$ compare to the graph of $y = \sqrt{x}$?




The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$ and the range is $\{y \mid y \geq -3, y \in \mathrm{R}\}$.

## Graphing Radical Functions Using Transformations

You can graph a radical function of the form $y = a\sqrt{b(x - h)} + k$ by transforming the graph of $y = \sqrt{x}$ based on the values of $a$, $b$, $h$, and $k$. The effects of changing parameters in radical functions are the same as the effects of changing parameters in other types of functions.

- Parameter $a$ results in a vertical stretch of the graph of $y = \sqrt{x}$ by a factor of $|a|$. If $a < 0$, the graph of $y = \sqrt{x}$ is reflected in the $x$-axis.
- Parameter $b$ results in a horizontal stretch of the graph of $y = \sqrt{x}$ by a factor of $\frac{1}{|b|}$. If $b < 0$, the graph of $y = \sqrt{x}$ is reflected in the $y$-axis.
- Parameter $h$ determines the horizontal translation. If $h > 0$, the graph of $y = \sqrt{x}$ is translated to the right $h$ units. If $h < 0$, the graph is translated to the left $|h|$ units.
- Parameter $k$ determines the vertical translation. If $k > 0$, the graph of $y = \sqrt{x}$ is translated up $k$ units. If $k < 0$, the graph is translated down $|k|$ units.

## Example 2

### Graph Radical Functions Using Transformations

$y = a\sqrt{b(x-h)} + k$

Sketch the graph of each function using transformations. Compare the domain and range to those of $y = \sqrt{x}$ and identify any changes.

**a)** $y = 3\sqrt{-(x - 1)}$ **b)** $y - 3 = -\sqrt{2x}$

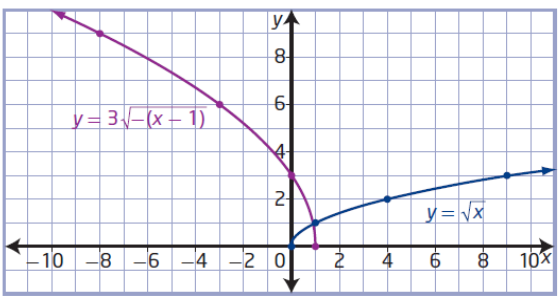
**a)** $y = 3\sqrt{-(x-1)}$

$a=3$ $b=-1$ $h=1$ $k=0$

| x | y |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |
| 25 | 5 |

$(x,y) \longrightarrow [-1x+1, 3y+0]$

| x | y |
|---|---|
| 1 | 0 |
| 0 | 3 |
| -3 | 6 |
| -8 | 9 |
| -15 | 12 |
| -24 | 15 |




Domain:

$\{x \mid x \leq 1, x \in R\}$

$-(x-1) \geq 0$

$-x+1 \geq 0$

$-x \geq -1$

$x \leq 1$

Range:

$\{y \mid y \geq 0, y \in R\}$

**b)** $y - 3 = -\sqrt{2x}$

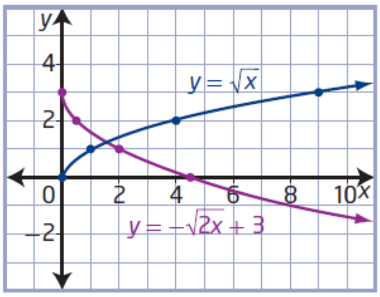
$y = -\sqrt{2x} + 3$

$a = -1 \quad b = 2 \quad h = 0 \quad k = 3$

| $x$ | $y$ |
|---|---|
| 0 | 0 |
| 1 | 1 |
| 4 | 2 |
| 9 | 3 |
| 16 | 4 |
| 25 | 5 |

$(x, y) \longrightarrow \left[\frac{x}{2} + 0, -1y + 3\right]$

| $x$ | $y$ |
|---|---|
| 0 | 3 |
| 0.5 | 2 |
| 2 | 1 |
| 4.5 | 0 |
| 8 | -1 |
| 12.5 | -2 |



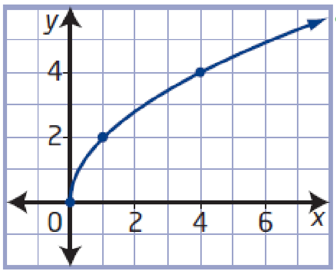

Domain:
$\{x \mid x \geq 0, x \in R\}$

Range:
$\{y \mid y \leq 3, y \in R\}$

## Example 3

### Determine a Radical Function From a Graph

Mayleen is designing a symmetrical pattern. She sketches the curve shown and wants to determine its equation and the equation of its reflection in each quadrant. The graph is a transformation of the graph of $y = \sqrt{x}$. What are the equations of the four functions Mayleen needs to work with?



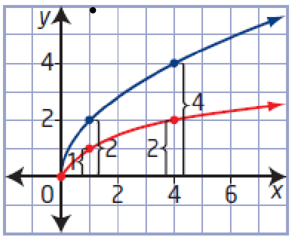

A radical function that involves a stretch can be obtained from either a vertical stretch or a horizontal stretch. Use an equation of the form $y = a\sqrt{x}$ or $y = \sqrt{bx}$ to represent the image function for each type of stretch.

**Method 1: Compare Vertical or Horizontal Distances**

Superimpose the graph of $y = \sqrt{x}$ and compare corresponding distances to determine the factor by which the function has been stretched.

**View as a Vertical Stretch ($y = a\sqrt{x}$)**

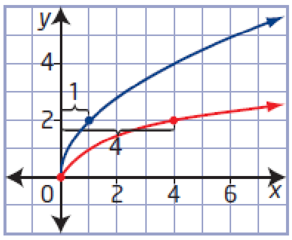
Each vertical distance is 2 times the corresponding distance for $y = \sqrt{x}$.




This represents a vertical stretch by a factor of 2, which means $a = 2$. The equation $y = 2\sqrt{x}$ represents the function.

**View as a Horizontal Stretch ($y = \sqrt{bx}$)**

Each horizontal distance is $\frac{1}{4}$ the corresponding distance for $y = \sqrt{x}$.




This represents a horizontal stretch by a factor of $\frac{1}{4}$, which means $b = 4$. The equation $y = \sqrt{4x}$ represents the function.

Express the equation of the function as either $y = 2\sqrt{x}$ or $y = \sqrt{4x}$.

$y = 2\sqrt{x}$ → quadrant 1

$y = 2\sqrt{-x}$ → quadrant 2

$y = -2\sqrt{-x}$ → quadrant 3

$y = -2\sqrt{x}$ → quadrant 4

## Example 4

### Model the Speed of Sound

Justin's physics textbook states that the speed, $s$, in metres per second, of sound in dry air is related to the air temperature, $T$, in degrees Celsius, by the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$.

**a)** Determine the domain and range in this context.

**b)** On the Internet, Justin finds another formula for the speed of sound, $s = 20\sqrt{T + 273}$. Use algebra to show that the two functions are approximately equivalent.

**c)** How is the graph of this function related to the graph of the base square root function? Which transformation do you predict will be the most noticeable on a graph?

**d)** Graph the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$ using technology.

**e)** Determine the speed of sound, to the nearest metre per second, at each of the following temperatures.

- **i)** 20 °C (normal room temperature)
- **ii)** 0 °C (freezing point of water)
- **iii)** −63 °C (coldest temperature ever recorded in Canada)
- **iv)** −89 °C (coldest temperature ever recorded on Earth)

a) Domain: $1 + \frac{T}{273.15} \geq 0$ $\{T \mid T \geq -273.15, T \in R\}$

$\frac{T}{273.15} \geq -1$

$T \geq -273.15°$

Range: $\{s \mid s \geq 0, s \in R\}$

b) $s = 331.3\sqrt{\frac{1}{1} + \frac{T}{273.15}}$

$s = 331.3\sqrt{\frac{273.15}{273.15} + \frac{T}{273.15}}$

$s = 331.3\sqrt{\frac{273.15 + T}{273.15}}$

$s = 331.3\frac{\sqrt{273.15 + T}}{\sqrt{273.15}}$

$s = 331.3\frac{\sqrt{273.15 + T}}{16.527}$

$s = 20.04\sqrt{273.15 + T}$

$s \approx 20\sqrt{T + 273}$

The graph of $s = \sqrt{T}$ is stretched vertically by a factor of about 20 and then translated about 273 units to the left. Translating 273 units to the left will be most noticeable on the graph of the function.

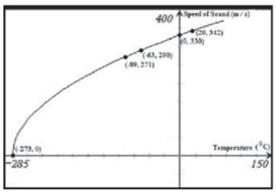
Are these transformations consistent with the domain and range?

c) $s = 20\sqrt{T + 273}$

$a = 20 \rightarrow$ vertical stretch by a factor of 20

$h = -273 \rightarrow$ translation of 273 units to the left

d)



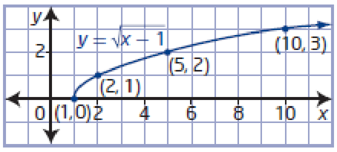

Are your answers to part c) confirmed by the graph?

e)

| | Temperature (°C) | Approximate Speed of Sound (m/s) |
|---|---|---|
| i) | 20 | 343 |
| ii) | 0 | 331 |
| iii) | −63 | 291 |
| iv) | −89 | 272 |

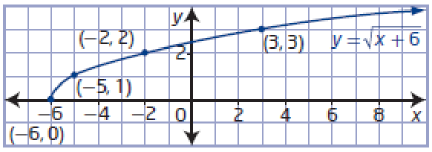
**2.1 Radical Functions and Transformations, pages 72 to 77**

**1. a)**



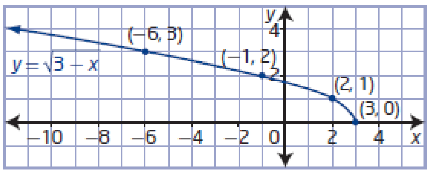

domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

**b)**



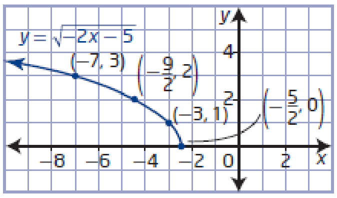

domain $\{x \mid x \geq -6, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

**c)**




domain $\{x \mid x \leq 3, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

**d)**




domain $\left\{x \mid x \leq -\frac{5}{2}, x \in \mathrm{R}\right\}$,
range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

**2. a)** $a = 7 \rightarrow$ vertical stretch by a factor of 7
$h = 9 \rightarrow$ horizontal translation 9 units right
domain $\{x \mid x \geq 9, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

**b)** $b = -1 \rightarrow$ reflected in $y$-axis
$k = 8 \rightarrow$ vertical translation up 8 units
domain $\{x \mid x \leq 0, x \in \mathrm{R}\}$, range $\{y \mid y \geq 8, y \in \mathrm{R}\}$

**c)** $a = -1 \rightarrow$ reflected in $x$-axis
$b = \frac{1}{5} \rightarrow$ horizontal stretch factor of 5
domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$, range $\{y \mid y \leq 0, y \in \mathrm{R}\}$

**d)** $a = \frac{1}{3} \rightarrow$ vertical stretch factor of $\frac{1}{3}$
$h = -6 \rightarrow$ horizontal translation 6 units left
$k = -4 \rightarrow$ vertical translation 4 units down
domain $\{x \mid x \geq -6, x \in \mathrm{R}\}$,
range $\{y \mid y \geq -4, y \in \mathrm{R}\}$

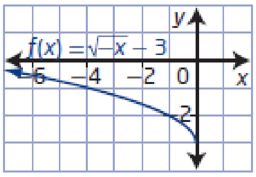
**3. a)** B **b)** A **c)** D **d)** C

**4. a)** $y = 4\sqrt{x + 6}$ **b)** $y = \sqrt{8x} - 5$

**c)** $y = \sqrt{-(x - 4)} + 11$ or $y = \sqrt{-x + 4} + 11$

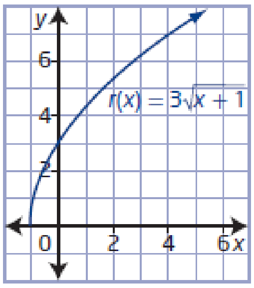
**d)** $y = -0.25\sqrt{0.1x}$ or $y = -\frac{1}{4}\sqrt{\frac{1}{10}x}$

**5. a)**



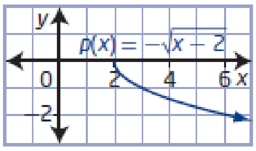

domain
$\{x \mid x \leq 0, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq -3, y \in \mathbb{R}\}$

**b)**



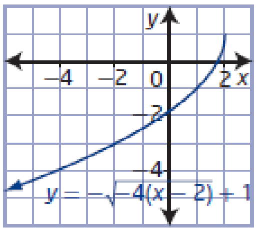

domain
$\{x \mid x \geq -1, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq 0, y \in \mathbb{R}\}$

**c)**



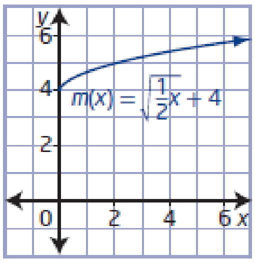

domain
$\{x \mid x \geq 2, x \in \mathbb{R}\}$,
range
$\{y \mid y \leq 0, y \in \mathbb{R}\}$

**d)**



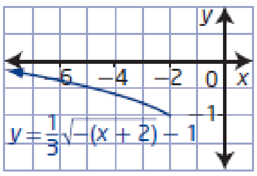

domain
$\{x \mid x \leq 2, x \in \mathbb{R}\}$,
range
$\{y \mid y \leq 1, y \in \mathbb{R}\}$

**e)**




domain
$\{x \mid x \geq 0, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq 4, y \in \mathbb{R}\}$

**f)**




domain
$\{x \mid x \leq -2, x \in \mathbb{R}\}$
range
$\{y \mid y \geq -1, y \in \mathbb{R}\}$

**6. a)** $a = \frac{1}{4} \rightarrow$ vertical stretch factor of $\frac{1}{4}$

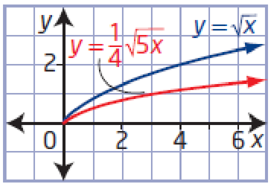
$b = 5 \rightarrow$ horizontal stretch factor of $\frac{1}{5}$

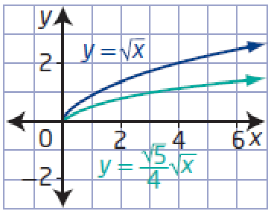
**b)** $y = \frac{\sqrt{5}}{4}\sqrt{x}$, $y = \sqrt{\frac{5}{16}x}$

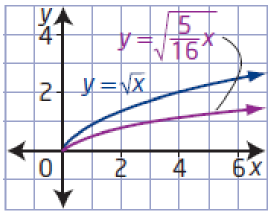
**c)** $a = \frac{\sqrt{5}}{4} \rightarrow$ vertical stretch factor of $\frac{\sqrt{5}}{4}$

$b = \frac{5}{16} \rightarrow$ horizontal stretch factor of $\frac{16}{5}$

**d)**



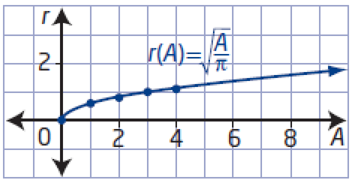




All graphs are the same.

**7. a)** $r(A) = \sqrt{\frac{A}{\pi}}$

**b)**

| A | r |
|---|---|
| 0 | 0 |
| 1 | 0.6 |
| 2 | 0.8 |
| 3 | 1.0 |
| 4 | 1.1 |




**8. a)** $b = 1.50 \rightarrow$ horizontal stretch factor of $\frac{1}{1.50}$ or $\frac{2}{3}$

**b)** $d \approx 1.22\sqrt{h}$ Example: I prefer the original function because the values are exact.

**c)** approximately 5.5 miles

**9. a)** domain $\{x \mid x \geq 0, x \in R\}$, range $\{y \mid y \geq -13, y \in R\}$

**b)** $h = 0 \rightarrow$ no horizontal translation

$k = 13 \rightarrow$ vertical translation down 13 units

**10. a)** $y = -\sqrt{x + 3} + 4$ **b)** $y = \frac{1}{2}\sqrt{x + 5} - 3$

**c)** $y = 2\sqrt{-(x - 5)} - 1$ or $y = 2\sqrt{-x + 5} - 1$

**d)** $y = -4\sqrt{-(x - 4)} + 5$ or $y = -4\sqrt{-x + 4} + 5$

**11.** Examples:

**a)** $y - 1 = \sqrt{x - 6}$ or $y = \sqrt{x - 6} + 1$

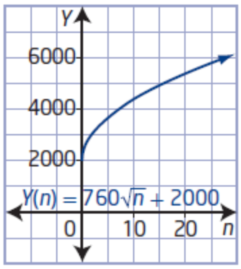
**b)** $y = -\sqrt{x + 7} - 9$ **c)** $y = 2\sqrt{-x + 4} - 3$

**d)** $y = -\sqrt{-(x + 5)} + 8$

**12. a)** $a = 760 \rightarrow$ vertical stretch factor of 760

$k = 2000 \rightarrow$ vertical translation up 2000

**b)**




**c)** domain

$\{n \mid n \geq 0, n \in R\}$

range

$\{Y \mid Y \geq 2000, Y \in R\}$

**d)** The minimum yield is 2000 kg/hectare. Example: The domain and range imply that the more nitrogen added, the greater the yield without end. This is not realistic.

# Square Root of a Function

## Focus on...

- sketching the graph of $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- explaining strategies for graphing $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- comparing the domains and ranges of the functions $y = f(x)$ and $y = \sqrt{f(x)}$, and explaining any differences

**square root of a function**

- the function $y = \sqrt{f(x)}$ is the square root of the function $y = f(x)$
- $y = \sqrt{f(x)}$ is only defined for $f(x) \geq 0$

The function $y = \sqrt{2x + 1}$ represents the **square root of the function** $y = 2x + 1$. (Linear)

| $x$ | $y = 2x + 1$ | $y = \sqrt{2x + 1}$ |
|---|---|---|
| 0 | 1 | 1 |
| 4 | 9 | 3 |
| 12 | 25 | 5 |
| 24 | 49 | 7 |
| ⋮ | ⋮ | ⋮ |

D: $2x+1 \geq 0$

$2x \geq -1$

$x \geq -\frac{1}{2}$

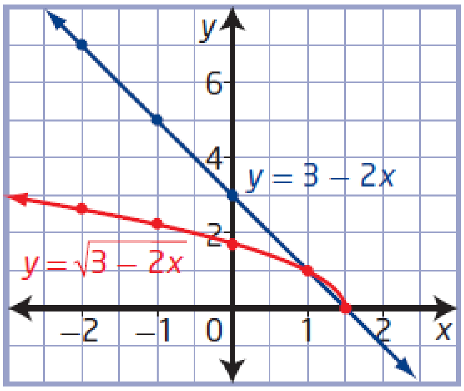
## Example 1

### Compare Graphs of a Linear Function and the Square Root of the Function

**a)** Given $f(x) = 3 - 2x$, graph the functions $y = f(x)$ and $y = \sqrt{f(x)}$.
**b)** Compare the two functions.

Use a table of values to graph $y = 3 - 2x$ and $y = \sqrt{3 - 2x}$.

| $x$ | $y = 3 - 2x$ | $y = \sqrt{3 - 2x}$ |
|---|---|---|
| –2 | 7 | $\sqrt{7}$ |
| –1 | 5 | $\sqrt{5}$ |
| 0 | 3 | $\sqrt{3}$ |
| 1 | 1 | 1 |
| 1.5 | 0 | 0 |
| 2 | -1 | undefined |




For $y = 3 - 2x$:

D: $\{x \mid x \in \mathbb{R}\}$

R: $\{y \mid y \in \mathbb{R}\}$

For $y = \sqrt{3-2x}$

$3 - 2x \geq 0$

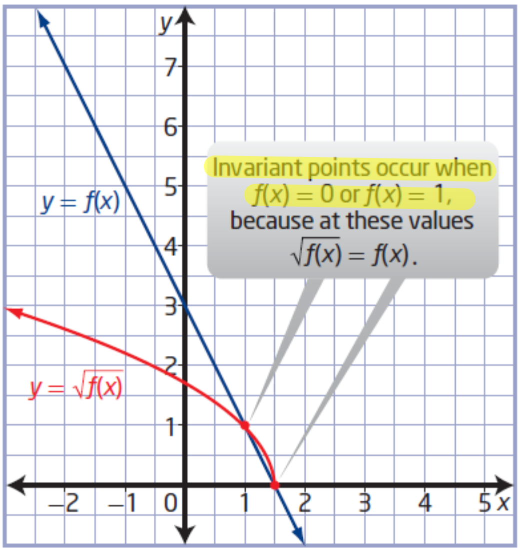
$-2x \geq -3$

$x \leq 1.5$

D: $\{x \mid x \leq 1.5, x \in \mathbb{R}\}$

R: $\{y \mid y \geq 0, y \in \mathbb{R}\}$

**b)** Compare the graphs.




Why is the graph of $y = \sqrt{f(x)}$ above the graph of $y = f(x)$ for values of $y$ between 0 and 1? Will this always be true? Yes $\rightarrow$ $y = 0.25$

$y = \sqrt{0.25} = 0.5$

For $y = f(x)$, the domain is $\{x \mid x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$.
For $y = \sqrt{f(x)}$, the domain is $\{x \mid x \leq 1.5,\ x \in \mathrm{R}\}$ and the range is $\{y \mid y \geq 0,\ y \in \mathrm{R}\}$.

Invariant points occur at (1, 1) and (1.5, 0).

How does the domain of the graph of $y = \sqrt{f(x)}$ relate to the restrictions on the variable in the radicand? How could you determine the domain algebraically?

## Relative Locations of $y = f(x)$ and $y = \sqrt{f(x)}$

The domain of $y = \sqrt{f(x)}$ consists only of the values in the domain of $f(x)$ for which $f(x) \geq 0$.

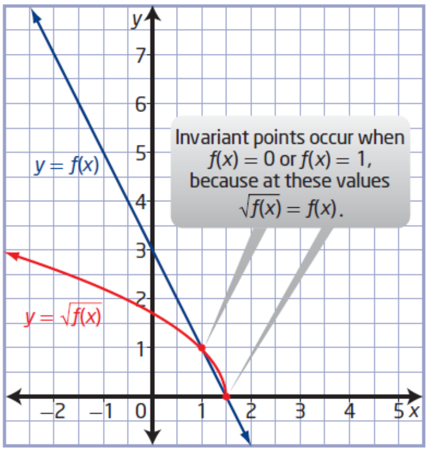
The range of $y = \sqrt{f(x)}$ consists of the square roots of the values in the range of $y = f(x)$ for which $\sqrt{f(x)}$ is defined.

The graph of $y = \sqrt{f(x)}$ exists only where $f(x) \geq 0$. You can predict the location of $y = \sqrt{f(x)}$ relative to $y = f(x)$ using the values of $f(x)$.

Inv. Pt. Inv. Pt

| **Value of $f(x)$** | $f(x) < 0$ | $f(x) = 0$ | $0 < f(x) < 1$ | $f(x) = 1$ | $f(x) > 1$ |
|---|---|---|---|---|---|
| **Relative Location of Graph of $y = \sqrt{f(x)}$** | The graph of $y = \sqrt{f(x)}$ is undefined. | The graphs of $y = \sqrt{f(x)}$ and $y = f(x)$ intersect on the $x$-axis. | The graph of $y = \sqrt{f(x)}$ is above the graph of $y = f(x)$. | The graph of $y = \sqrt{f(x)}$ intersects the graph of $y = f(x)$. | The graph of $y = \sqrt{f(x)}$ is below the graph of $y = f(x)$. |

**b)** Compare the graphs.

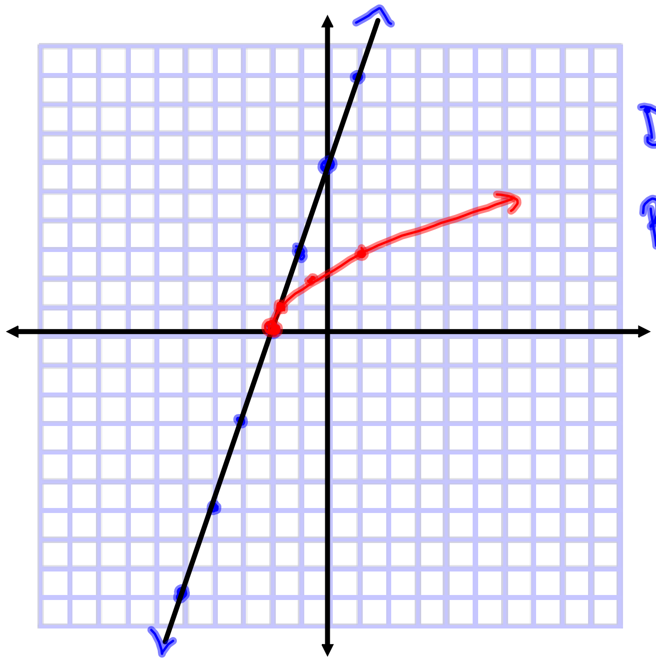



Why is the graph of $y = \sqrt{f(x)}$ above the graph of $y = f(x)$ for values of $y$ between 0 and 1? Will this always be true?

## Your Turn

**a)** Given $g(x) = 3x + 6$, graph the functions $y = g(x)$ and $y = \sqrt{g(x)}$.

**b)** Identify the domain and range of each function and any invariant points.



$y = 3x+6$

D: $\{x \mid x \in R\}$

R: $\{y \mid y \in R\}$

$y = \sqrt{3x+6}$

D: $\{x \mid x \geq -2, x \in R\}$

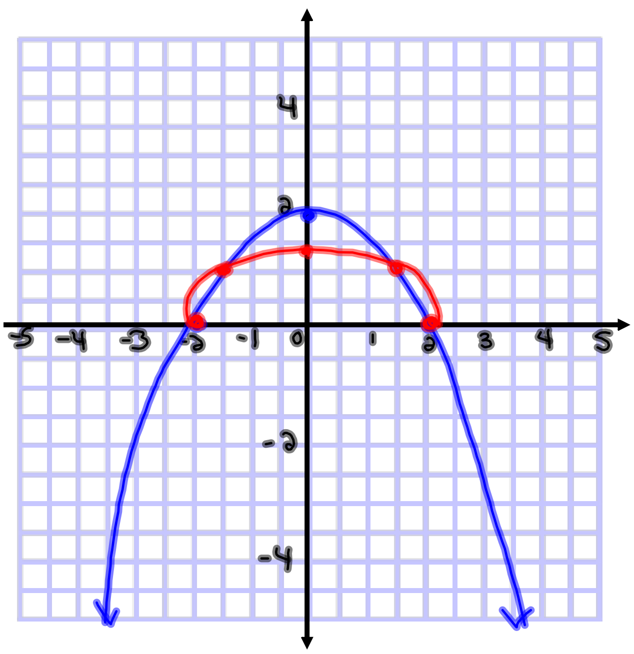
R: $\{y \mid y \geq 0, y \in R\}$

## Example 2

### Compare the Domains and Ranges of $y = f(x)$ and $y = \sqrt{f(x)}$

Identify and compare the domains and ranges of the functions in each pair.

**a)** $y = 2 - 0.5x^2$ and $y = \sqrt{2 - 0.5x^2}$




$y=2-0.5x^2$

| x | y |
|---|---|
| -4 | -6 |
| -2 | 0 |
| 0 | 2 |
| 2 | 0 |
| 4 | -6 |

D: $\{x \mid x \in R\}$

R: $\{y \mid y \leq 2, y \in R\}$

$y=\sqrt{2-0.5x^2}$

| x | y |
|---|---|
| -4 | und. |
| -2 | 0 |
| 0 | $\sqrt{2}$ |
| 2 | 0 |
| 4 | und |

D: $\{x \mid -2 \leq x \leq 2, x \in R\}$

R: $\{y \mid 0 \leq y \leq \sqrt{2}, y \in R\}$

## Example 3

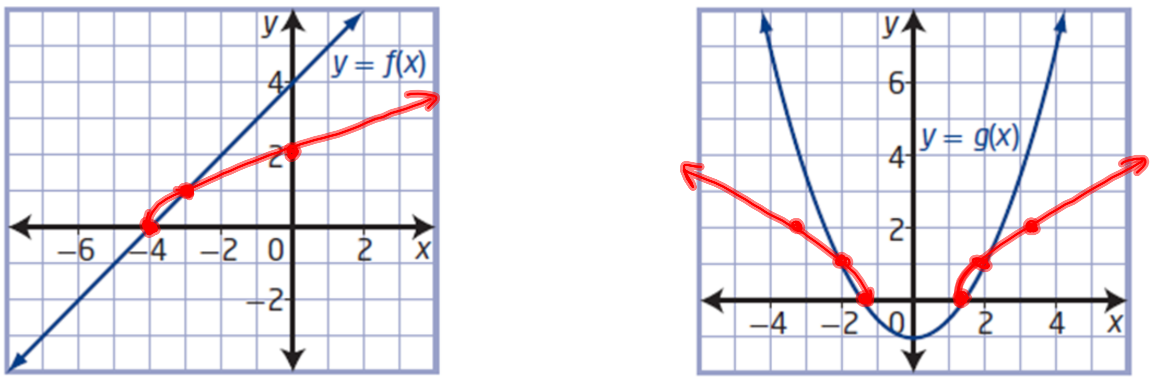
### Graph the Square Root of a Function From the Graph of the Function

**Step 1:** Locate invariant points $f(x)=0$ and $f(x)=1$

**Step 2:** Draw the portion of each graph between the invariant points

**Step 3:** Locate other key points on $y = f(x)$ and $y = g(x)$ where the values are greater than 1. Transform these points to locate image points on the graphs of $y = \sqrt{f(x)}$ and $y = \sqrt{g(x)}$.

Using the graphs of $y = f(x)$ and $y = g(x)$, sketch the graphs of $y = \sqrt{f(x)}$ and $y = \sqrt{g(x)}$.



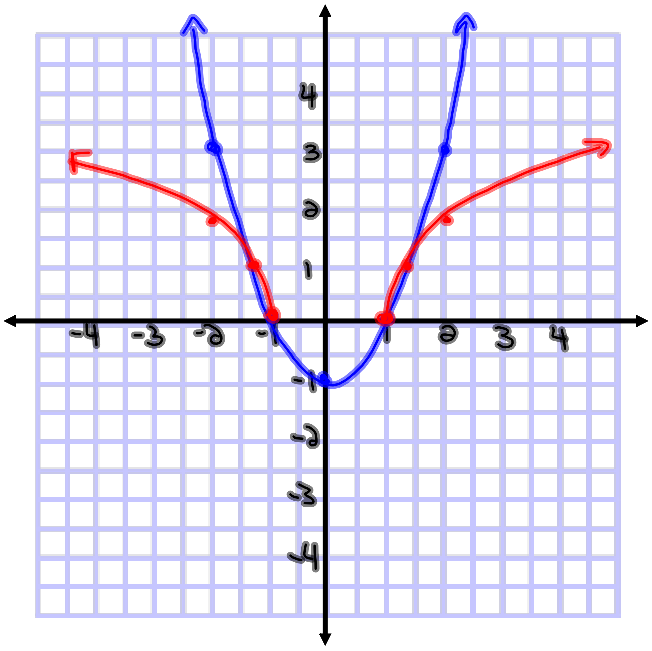
## Key Ideas

- You can use values of $f(x)$ to predict values of $\sqrt{f(x)}$ and to sketch the graph of $y = \sqrt{f(x)}$.
- The key values to consider are $f(x) = 0$ and $f(x) = 1$.
- The domain of $y = \sqrt{f(x)}$ consists of all values in the domain of $f(x)$ for which $f(x) \geq 0$.
- The range of $y = \sqrt{f(x)}$ consists of the square roots of all values in the range of $f(x)$ for which $f(x)$ is defined.
- The $y$-coordinates of the points on the graph of $y = \sqrt{f(x)}$ are the square roots of the $y$-coordinates of the corresponding points on the original function $y = f(x)$.

What do you know about the graph of $y = \sqrt{f(x)}$ at $f(x) = 0$ and $f(x) = 1$? How do the graphs of $y = f(x)$ and $y = \sqrt{f(x)}$ compare on either side of these locations?

## Your Turn

1) Identify and compare the domains and ranges of the functions $y = x^2 - 1$ and $y = \sqrt{x^2 - 1}$. Verify your answers.




$y = x^2 - 1$ D: $\{x \mid x \in \mathbb{R}\}$

R: $\{y \mid y \geq -1, y \in \mathbb{R}\}$

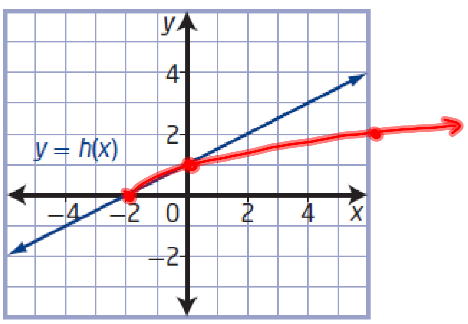
| x | y |
|---|---|
| -2 | 3 |
| -1 | 0 |
| 0 | -1 |
| 1 | 0 |
| 2 | 3 |

$y = \sqrt{x^2 - 1}$ D: $\{x \mid x \leq -1 \text{ and } x \geq 1, x \in \mathbb{R}\}$

R: $\{y \mid y \geq 0, y \in \mathbb{R}\}$

| x | y |
|---|---|
| -2 | $\sqrt{3}$ |
| -1 | 0 |
| 0 | undef. |
| 1 | 0 |
| 2 | $\sqrt{3}$ |

2) Using the graph of $y = h(x)$, sketch the graph of $y = \sqrt{h(x)}$.



# Solving Radical Equations Graphically & Algebraically

**Focus on…**

- relating the roots of radical equations and the $x$-intercepts of the graphs of radical functions (same)
- determining approximate solutions of radical equations graphically

## Example 1

### Relate Roots and *x*-Intercepts

**a)** Determine the root(s) of $\sqrt{x+5} - 3 = 0$ algebraically.

**b)** Using a graph, determine the x-intercept(s) of the graph of $y = \sqrt{x+5} - 3$. (y=0)

**c)** Describe the connection between the root(s) of the equation and the *x*-intercept(s) of the graph of the function.

a) $\sqrt{x+5} - 3 = 0$

$\sqrt{x+5} = 3$

$x+5 = 9$

$x = 4$

b) $y = \sqrt{x+5} - 3$

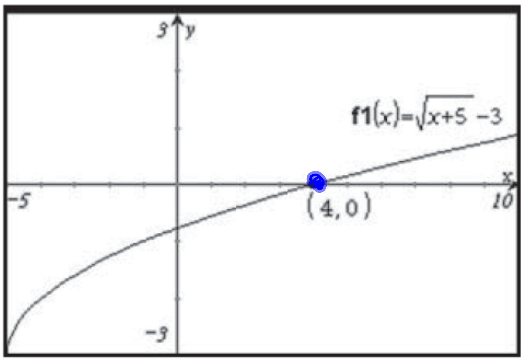
$0 = \sqrt{x+5} - 3$

$3 = \sqrt{x+5}$

$9 = x+5$

$4 = x$  (4,0)

c) They are the same



## Example 2

### Solve a Radical Equation Involving an Extraneous Solution

(isolate the radical)

Solve the equation $\sqrt{x + 5} = x + 3$ algebraically and graphically.

$\sqrt{x+5} = x+3$

$x+5 = x^2+6x+9$ (Square both sides)

$0 = x^2+5x+4$ (Bring everything to one side)

$0 = (x+1)(x+4)$ (Simple Trinomial)

| $x+1=0$ | $x+4=0$ |
|---|---|
| $x=-1$ | $x=-4$ ← extraneous root |

Test $x=-1$

| L.S. | R.S |
|---|---|
| $\sqrt{-1+5}$ | $-1+3$ |
| $\sqrt{4}$ | $2$ |
| $2$ | ✓ |

Test $x=-4$

| L.S. | R.S. |
|---|---|
| $\sqrt{-4+5}$ | $-4+3$ |
| $\sqrt{1}$ | $-1$ |
| $1$ | ✗ |

## Example 3

### Approximate Solutions to Radical Equations (isolate the radical)

**a)** Solve the equation $\sqrt{3x^2 - 5} = x + 4$ graphically. Express your answer to the nearest tenth.

**b)** Verify your solution algebraically.

$$\sqrt{3x^2-5} = x+4$$

$$3x^2-5 = x^2+8x+16$$ (Square both sides)

$$2x^2-8x-21=0$$ (Bring everything to one side)

$a=2$ $b=-8$ $c=-21$

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}$$ (Use Quadratic Formula)

$$x=\frac{-(-8)\pm\sqrt{(-8)^2-4(2)(-21)}}{2(2)}$$

$$x=\frac{8\pm\sqrt{64+168}}{4}$$

$$x=\frac{8\pm\sqrt{232}}{4}$$

$$x=\frac{8\pm 15.23}{4}$$

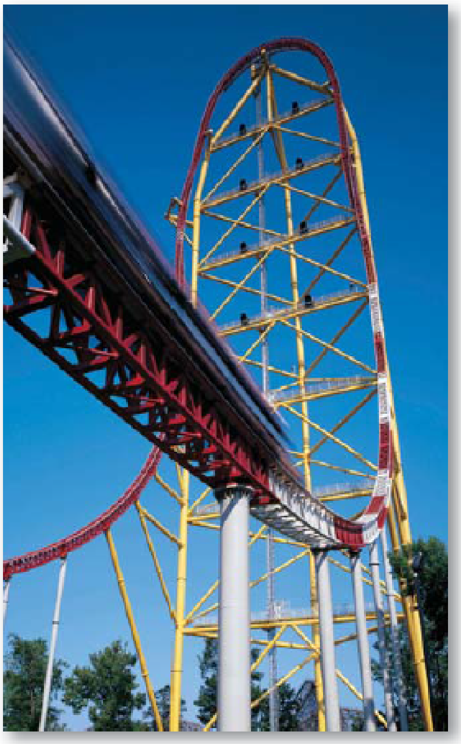
$\boxed{x\approx 5.8}$ | $\boxed{x\approx -1.8}$

Test your answers (both are solutions)

## Example 4

### Solve a Problem Involving a Radical Equation

An engineer designs a roller coaster that involves a vertical drop section just below the top of the ride. She uses the equation $v = \sqrt{(v_0)^2 + 2ad}$ to model the velocity, $v$, in feet per second, of the ride's cars after dropping a distance, $d$, in feet, with an initial velocity, $v_0$, in feet per second, at the top of the drop, and constant acceleration, $a$, in feet per second squared. The design specifies that the speed of the ride's cars be 120 ft/s at the bottom of the vertical drop section. If the initial velocity of the coaster at the top of the drop is 10 ft/s and the only acceleration is due to gravity, 32 ft/s², what vertical drop distance should be used, to the nearest foot?



Given:

$V = 120$

$V_0 = 10$

$a = 32$

$$V = \sqrt{(V_0)^2 + 2ad}$$

$$120 = \sqrt{(10)^2 + 2(32)d}$$

$$120 = \sqrt{100 + 64d}$$

$$14400 = 100 + 64d$$

$$14300 = 64d$$

$$\boxed{223.4 \text{ ft} = d}$$

## Key Ideas

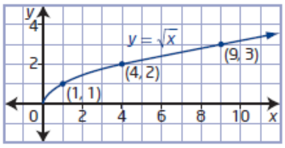
- You can solve radical equations algebraically and graphically.
- The solutions or roots of a radical equation are equivalent to the $x$-intercepts of the graph of the corresponding radical function. You can use either of the following methods to solve radical equations graphically:
  - Graph the corresponding function and identify the value(s) of the $x$-intercept(s).
  - Graph the system of functions that corresponds to the expression on each side of the equal sign, and then identify the value(s) of $x$ at the point(s) of intersection.

Homework

Finish Chapter Review

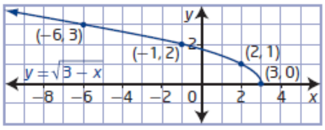
### Chapter 2 Review, pages 99 to 101

**1. a)**



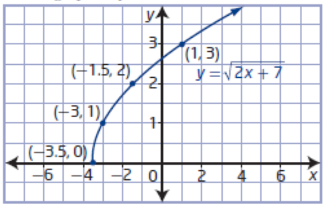

domain $\{x \mid x \geq 0, x \in R\}$
range $\{y \mid y \geq 0, y \in R\}$ All values in the table lie on the smooth curve graph of $y = \sqrt{x}$.

**b)**




domain $\{x \mid x \leq 3, x \in R\}$
range $\{y \mid y \geq 0, y \in R\}$ All points in the table lie on the graph of $y = \sqrt{3 - x}$.

**c)**




domain $\{x \mid x \geq -3.5, x \in R\}$
range $\{y \mid y \geq 0, y \in R\}$ All points in the table lie on the graph of $y = \sqrt{2x + 7}$.

**2.** Use $y = a\sqrt{b(x - h)} + k$ to describe transformations.

**a)** $a = 5 \rightarrow$ vertical stretch factor of 5
$h = -20 \rightarrow$ horizontal translation left 20 units;
domain $\{x \mid x \geq -20, x \in R\}$; range $\{y \mid y \geq 0, y \in R\}$

**b)** $b = -2 \rightarrow$ horizontal stretch factor of $\frac{1}{2}$, then reflected on $y$-axis: $k = -8 \rightarrow$ vertical translation of 8 units down.
domain $\{x \mid x \leq 0, x \in R\}$; range $\{y \mid y \geq -8, y \in R\}$

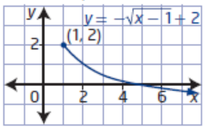
**c)** $a = -1 \rightarrow$ reflect in $x$-axis
$b = \frac{1}{6} \rightarrow$ horizontal stretch factor of 6
$h = 11 \rightarrow$ horizontal translation right 11 units;
domain $\{x \mid x \geq 11, x \in R\}$, range $\{y \mid y \leq 0, y \in R\}$.

**3. a)** $y = \sqrt{\frac{1}{10}x} + 12$, domain $\{x \mid x \geq 0, x \in R\}$, range $\{y \mid y \geq 12, y \in R\}$

**b)** $y = -2.5\sqrt{x + 9}$
domain $\{x \mid x \geq -9, x \in R\}$, range $\{y \mid y \leq 0, y \in R\}$

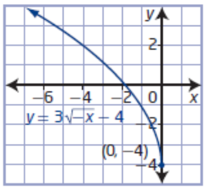
**c)** $y = \frac{1}{20}\sqrt{-\frac{2}{5}(x - 7)} - 3$,
domain $\{x \mid x \leq 7, x \in R\}$, range $\{y \mid y \geq -3, y \in R\}$

**4. a)**



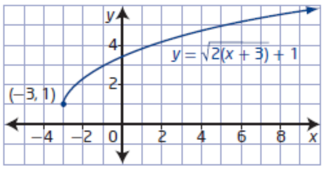

domain $\{x \mid x \geq 1, x \in R\}$, range $\{y \mid y \leq 2, y \in R\}$

**b)**




domain $\{x \mid x \leq 0, x \in R\}$, range $\{y \mid y \geq -4, y \in R\}$

**c)**

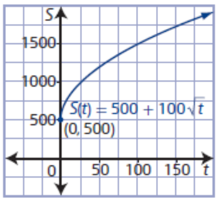



domain $\{x \mid x \geq -3, x \in R\}$, range $\{y \mid y \geq 1, y \in R\}$

**5.** The domain is affected by a horizontal translation of 4 units right and by no reflection on the $y$-axis. The domain will have values of $x$ greater than or equal to 4, due to a translation of the graph 4 units right. The range is affected by vertical translation of 9 units up and a reflection on the $x$-axis. The range will be less than or equal to 9, because the graph has been moved up 9 units and reflected on the $x$-axis, therefore the range is less than or equal to 9, instead of greater than or equal to 9.

**6. a)** Given the general equation $y = a\sqrt{b(x - h)} + k$ to describe transformations, $a = 100$ indicates a vertical stretch by a factor of 100, $k = 500$ indicates a vertical translation up 500 units.

**b)**




Since the minimum value of the graph is 500, the minimum estimated sales will be 500 units.

**c)** domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$ The domain means that time is positive in this situation.
range $\{S(t) \mid S(t) \geq 500, S(t) \in \mathrm{W}\}$. The range means that the minimum sales are 500 units.

**d)** about 1274 units

**7. a)** $y = \sqrt{\frac{1}{4}(x + 3)} + 2$ **b)** $y = -2\sqrt{x + 4} + 3$

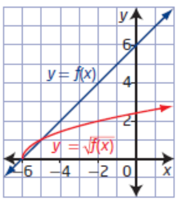
**c)** $y = 4\sqrt{-(x - 6)} - 4$

**8. a)** For $y = x - 2$, domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$; for $y = \sqrt{x - 2}$, domain $\{x \mid x \geq 2, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$. The domain changes because the square root function has restrictions. The range changes because the function only exists on or above the $x$-axis.

**b)** For $y = 10 - x$, domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$; for $y = \sqrt{10 - x}$, domain $\{x \mid x \leq 10, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$ The domain changes because the square root function has restrictions. The range changes because the function only exists on or above the $x$-axis.

**c)** For $y = 4x + 11$, domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$; for $y = \sqrt{4x + 11}$, domain $\left\{x \mid x \geq -\frac{11}{4}, x \in \mathrm{R}\right\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$. The domain changes because the square root function has restrictions. The range changes because the function only exists on or above the $x$-axis.

**9. a)**




Plot invariant points at the intersection of the graph and lines $y = 0$ and $y = 1$. Plot any points $(x, \sqrt{y})$ where the value of $y$ is a perfect square. Sketch a smooth curve through the invariant points and points satisfying $(x, \sqrt{y})$.

**b)** $y = \sqrt{f(x)}$ is positive when $f(x) > 0$,
$y = \sqrt{f(x)}$ does not exist when $f(x) < 0$.
$\sqrt{f(x)} > f(x)$ when $0 < f(x) < 1$ and
$f(x) > \sqrt{f(x)}$ when $f(x) > 1$

**c)** For $f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$; for $\sqrt{f(x)}$, domain $\{x \mid x \geq -6, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$, since $\sqrt{f(x)}$ is undefined when $f(x) < 0$.

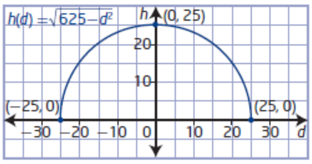
**10. a)** $y = 4 - x^2 \rightarrow$ domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \leq 4, y \in \mathrm{R}\}$ for $y = \sqrt{4 - x^2} \rightarrow$ domain $\{x \mid -2 \leq x \leq 2, x \in \mathrm{R}\}$, range $\{y \mid 0 \leq y \leq 2, y \in \mathrm{R}\}$, since $4 - x^2 > 0$ only between $-2$ and 2 then the domain of $y = \sqrt{4 - x^2}$ is $-2 \leq x \leq 2$. In the domain of $-2 \leq x \leq 2$ the maximum value of $y = 4 - x^2$ is 4, so the maximum value of $y = \sqrt{4 - x^2}$ is $\sqrt{4} = 2$ then the range of the function $y = \sqrt{4 - x^2}$ will be $0 \leq y \leq 2$.

**b)** $y = 2x^2 + 24 \rightarrow$ domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \geq 24, y \in \mathrm{R}\}$ for $y = \sqrt{2x^2 + 24} \rightarrow$ domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \geq \sqrt{24}, y \in \mathrm{R}\}$. The domain does not change since the entire graph of $y = 2x^2 + 24$ is above the $x$-axis. The range changes since the entire graph moves up 24 units and the graph itself opens up, so the range becomes $y \geq \sqrt{24}$.

**c)** $y = x^2 - 6x \rightarrow$ domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \geq -9, y \in \mathrm{R}\}$ for $y = \sqrt{x^2 - 6x} \rightarrow$ domain $\{x \mid x \leq 0 \text{ or } x \geq 6, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$, since $x^2 - 6x < 0$ between 0 and 6, then the domain is undefined in the interval $(0, 6)$ and exists when $x \leq 0$ or $x \geq 6$. The range changes because the function only exists above the $x$-axis.

**11. a)** $h(d) = \sqrt{625 - d^2}$

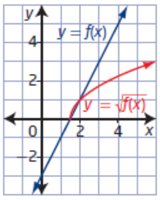
**b)**



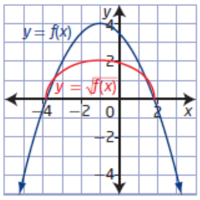

domain $\{d \mid -25 \leq d \leq 25, d \in \mathrm{R}\}$
range $\{h \mid 0 \leq h \leq 25, h \in \mathrm{R}\}$

**c)** In this situation, the values of $h$ and $d$ must be positive to express a positive distance. Therefore the domain changes to $\{d \mid 0 \leq d \leq 25, d \in \mathrm{R}\}$. Since the range of the original function $h(d) = \sqrt{625 - d^2}$ is always positive then the range does not change.

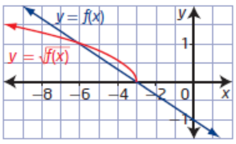
**12. a)**




**b)**

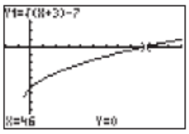



**c)**



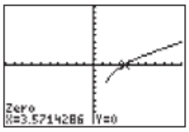
**13. a)** $x = 46$

**b)**

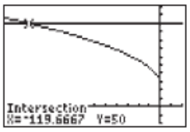



**c)** The root of the equation and the $x$ value of the $x$-intercept are the same.

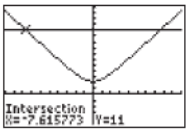
**14. a)** $x \approx 3.571$

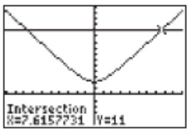



**b)** $x \approx -119.667$



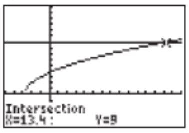

**c)** $x \approx -7.616$ and $x \approx 7.616$



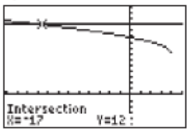




**15.** 4.13 m

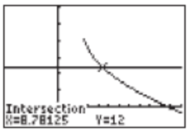
**16. a)** $x = 13.4$



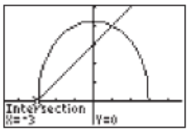

**b)** $x = -17$

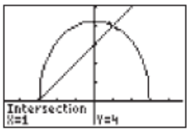



**c)** $x \approx 8.781$




**d)** $x = -3$ and 1




**17. a)** Jaime found two possible answers which are determined by solving a quadratic equation.

**b)** Carly found only one intersection at (5, 5) or $x$-intercept (5, 0) determined by possibly graphing.

**c)** Atid found an extraneous root of $x = 2$.

**18. a)** 130 m$^2$ **b)** 6 m