Angles and Angle Measure

- **What do you think about when you hear the word angle?**

Radian Measure

A radian is the angle subtended by an arc of length r (radius)

Link the Ideas

In the investigation, you encountered several key points associated with angle measure.

By convention, angles measured in a counterclockwise direction are said to be positive. Those measured in a clockwise direction are negative.

The angle AOB that you created measures 1 **radian**.

One full rotation is 360° or 2π radians.

One half rotation is 180° or π radians.

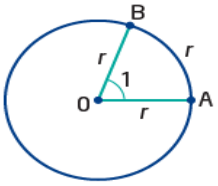
One quarter rotation is 90° or $\frac{\pi}{2}$ radians.

One eighth rotation is 45° or $\frac{\pi}{4}$ radians.

Many mathematicians omit units for radian measures. For example, $\frac{2\pi}{3}$ radians may be written as $\frac{2\pi}{3}$. Angle measures without units are considered to be in radians.

radian

- one radian is the measure of the central angle subtended in a circle by an arc equal in length to the radius of the circle
- $2\pi = 360°$ $= 1$ full rotation (or revolution)



$1° = \dfrac{\pi}{180}$ radians

$1 \text{ rad} = \dfrac{180}{\pi}$

Ex. Convert the following angles from radians to degrees:

a) π/6

$\frac{\pi}{6} \cdot \frac{180}{\pi}$ (30°)

$\frac{180\pi}{6\pi}$

30°

a) -2π/5

$\frac{-2\pi}{5} \cdot \frac{180}{\pi}$ (36)

−72°

c) 6.485

$6.485 \cdot \frac{180}{\pi}$

$\frac{1167.3}{\pi}$

371.6°

$1^\circ = \frac{\pi}{180}$ radians

$1 \text{ rad} = \frac{180}{\pi}$

Ex. Convert the following angles from degrees to radians:

a) 60°

$60^\circ \cdot \frac{\pi}{180}$

$\frac{60\pi}{180}$

$\frac{\pi}{3}$ ← exact value

1.05 ← approx.

b) 728°

$728 \cdot \frac{\pi}{180}$

$\frac{728\pi}{180}$

$\frac{182\pi}{45}$

c) -270°

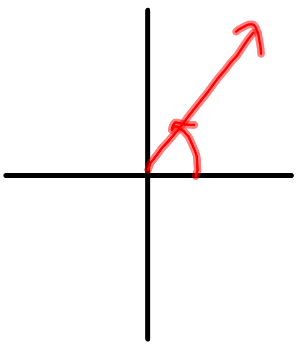
$-270^\circ \cdot \frac{\pi}{180}$

$-\frac{3\pi}{2}$

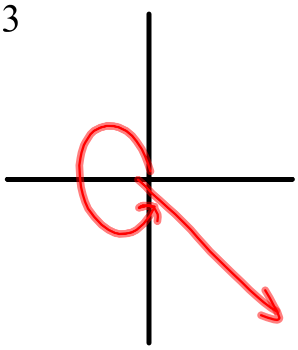
Sketching Angles

If the angle is positive rotate counterclockwise. If the angle is negative rotate clockwise. What do you notice about "a" and "c"?

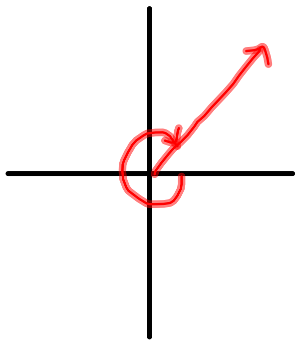
a) $50°$



b) $\frac{5\pi}{3}$



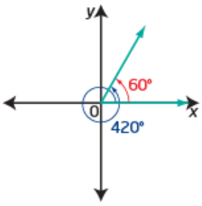
c) $-310°$



Coterminal Angles

When you sketch an angle of 60° and an angle of 420° in standard position, the terminal arms coincide. These are **coterminal angles**.

coterminal angles
- angles in standard position with the same terminal arms
- may be measured in degrees or radians
- $\frac{\pi}{4}$ and $\frac{9\pi}{4}$ are coterminal angles, as are 40° and −320°



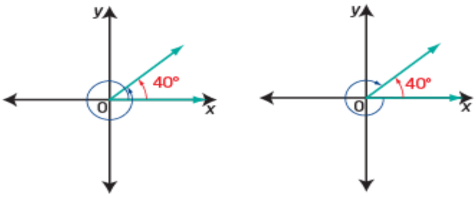
Identify Coterminal Angles

Determine one positive and one negative angle measure that is coterminal with each angle. In which quadrant does the terminal arm lie?

a) 40° **b)** −430° **c)** $\frac{8\pi}{3}$

Solution

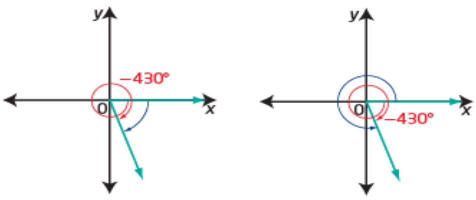
a) The terminal arm is in quadrant I.
To locate coterminal angles, begin on the terminal arm of the given angle and rotate in a positive or negative direction until the new terminal arm coincides with that of the original angle.



$40° + 360° = 400°$ $40° + (-360°) = -320°$
Two angles coterminal with 40° are 400° and −320°.

What other answers are possible?

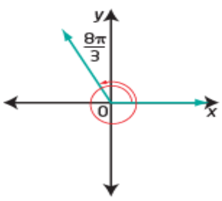
b) The terminal arm of −430° is in quadrant IV.



$-430° + 360° = -70°$ $-430° + 720° = 290°$
Two angles coterminal with −430° are 290° and −70°.

The reference angle is 70°.

c)



$\frac{8\pi}{3} = \frac{6\pi}{3} + \frac{2\pi}{3}$
So, the angle is one full rotation (2π) plus $\frac{2\pi}{3}$.

The terminal arm is in quadrant II.
There are 2π or $\frac{6\pi}{3}$ in one full rotation.
Counterclockwise one full rotation: $\frac{8\pi}{3} + \frac{6\pi}{3} = \frac{14\pi}{3}$
Clockwise one full rotation: $\frac{8\pi}{3} - \frac{6\pi}{3} = \frac{2\pi}{3}$
Clockwise two full rotations: $\frac{8\pi}{3} - \frac{12\pi}{3} = -\frac{4\pi}{3}$
Two angles coterminal with $\frac{8\pi}{3}$ are $\frac{2\pi}{3}$ and $-\frac{4\pi}{3}$.

Any given angle has an infinite number of angles coterminal with it, since each time you make one full rotation from the terminal arm, you arrive back at the same terminal arm. Angles coterminal with any angle θ can be described using the expression

$\theta \pm (360°)n$ or $\theta \pm 2\pi n$,

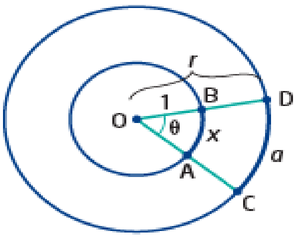
where n is a natural number. This way of expressing an answer is called the **general form**.

general form
- an expression containing parameters that can be given specific values to generate any answer that satisfies the given information or situation
- represents all possible cases

Arc Length of a Circle

All arcs that subtend a right angle $\left(\frac{\pi}{2}\right)$ have the same central angle, but they have different arc lengths depending on the radius of the circle. The arc length is proportional to the radius. This is true for any central angle and related arc length.

Consider two concentric circles with centre O. The radius of the smaller circle is 1, and the radius of the larger circle is r. A central angle of θ radians is subtended by arc AB on the smaller circle and arc CD on the larger one. You can write the following proportion, where x represents the arc length of the smaller circle and a is the arc length of the larger circle.



$$\frac{a}{x} = \frac{r}{1}$$
$$a = xr \qquad ①$$

Consider the circle with radius 1 and the sector with central angle θ. The ratio of the arc length to the circumference is equal to the ratio of the central angle to one full rotation.

$$\frac{x}{2\pi r} = \frac{\theta}{2\pi}$$

Why is $r = 1$?

$$x = \left(\frac{\theta}{2\pi}\right)2\pi(1)$$
$$x = \theta$$

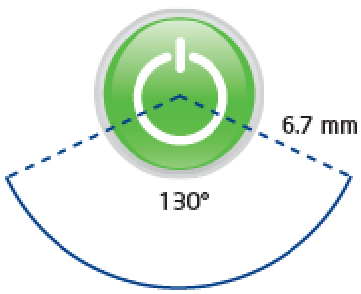
Substitute $x = \theta$ in ①.

$a = \theta r$

This formula, $a = \theta r$, works for any circle, provided that θ is measured in radians and both a and r are measured in the same units.

Determine Arc Length in a Circle

Rosemarie is taking a course in industrial engineering. For an assignment, she is designing the interface of a DVD player. In her plan, she includes a decorative arc below the on/off button. The arc has central angle 130° in a circle with radius 6.7 mm. Determine the length of the arc, to the nearest tenth of a millimetre.



Principal Angles

The smallest positive coterminal angle between 0 and 360° or 2π.

1) **Divide By 360° (how many rotations?).**
2) **Get rid of # of full rotations.**
3) **Mulitply decimal by 360° to find principal angle.**

Ex: 13784°

Key Ideas

- Angles can be measured using different units, including degrees and radians.
- An angle measured in one unit can be converted to the other unit using the relationships 1 full rotation = $360° = 2\pi$.
- An angle in standard position has its vertex at the origin and its initial arm along the positive x-axis.
- Angles that are coterminal have the same initial arm and the same terminal arm.
- An angle θ has an infinite number of angles that are coterminal expressed by $\theta \pm (360°)n$, $n \in N$, in degrees, or $\theta \pm 2\pi n$, $n \in N$, in radians.
- The formula $a = \theta r$, where a is the arc length; θ is the central angle, in radians; and r is the length of the radius, can be used to determine any of the variables given the other two, as long as a and r are in the same units.

Homework

Workbook #1-6 pages 114-117