Differentiation Rules

Product Rule:

The Product Rule If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)\frac{d}{dx}[g(x)] + g(x)\frac{d}{dx}[f(x)]$$

Express the product rule verbally if you are considering a function of the form...

f(x) = (First) X (Second)

In words, ***the Product Rule*** says that the ***derivative of a product of two functions is:*** ***the first function times the derivative of the second function, plus the derivative of the first function times the second function***

Get in the habit of verbalizing the rule as you differentiate...it will help when the functions get more complicated.

Differentiate the following function and simplify your answer:

$$h(t) = (t^3 - 5t)(6\sqrt{t} - t^{-5})$$

$$= (t^3 - 5t)(6t^{1/2} - t^{-5})$$

$$h'(t) = (t^3 - 5t)(3t^{-1/2} + 5t^{-6}) + (3t^2 - 5)(6t^{1/2} - t^{-5})$$

$$= 3t^{5/2} + 5t^{-3} - 15t^{1/2} - 25t^{-5} + 18t^{5/2} - 3t^{-3} - 30t^{1/2} + 5t^{-5}$$

$$= 21t^{5/2} - 45t^{1/2} + 2t^{-3} - 20t^{-5}$$

$$\boxed{= 21\sqrt{t^5} - 45\sqrt{t} + \frac{2}{t^3} - \frac{20}{t^5}}$$

$$f(x) = (7x^3 - x^2 + 5)(x^9 + 3x - 5)$$

⑤ $y = (2-\sqrt{x})(1+\sqrt{x}+3x)$; $(1,5)$

$= (2-x^{1/2})(1+x^{1/2}+3x)$

① $y' = (2-x^{1/2})\left(\frac{1}{2}x^{-1/2}+3\right) + \left(-\frac{1}{2}x^{-1/2}\right)(1+x^{1/2}+3x)$

② $y'(1) = (2-(1)^{1/2})\left(\frac{1}{2}(1)^{-1/2}+3\right) + \left(\frac{-1}{2}(1)^{-1/2}\right)(1+(1)^{1/2}+3(1))$

$y'(1) = (1)\left(\frac{7}{2}\right) + \left(-\frac{1}{2}\right)(5)$

$y'(1) = \frac{7}{2} - \frac{5}{2}$

$y'(1) = 1$ ← slope of the tangent "m"

③ $y - y_1 = m(x - x_1)$

$y - 5 = 1(x - 1)$

$y - 5 = x - 1$

$\boxed{0 = x - y + 4}$

Quotient Rule:

The Quotient Rule If f and g are differentiable, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{g(x)\frac{d}{dx}[f(x)] - f(x)\frac{d}{dx}[g(x)]}{[g(x)]^2}$$

Express the quotient rule verbally if you are considering a function of the form...

$$f(x) = \frac{(\text{First})}{(\text{Second})}$$

In words, ***the Quotient Rule*** says that the ***derivative of a quotient is:***
the denominator times the derivative of the numerator, minus the numerator times the derivative of the denominator, all divided by the square of the denominator.

Examples:

Differentiate the following functions and simplify your answers:

$$F(x)=\frac{x^2+2x-3}{x^3+1}$$

$$F'(x)=\frac{(x^3+1)(2x+2)-(x^2+2x-3)(3x^2)}{(x^3+1)^2}$$

$$=\frac{2x^4+2x^3+2x+2-(3x^4+6x^3-9x^2)}{(x^3+1)^2}$$

$$\boxed{=\frac{-x^4-4x^3+9x^2+2x+2}{(x^3+1)^2}}$$

$$F(x)=\frac{\sqrt{x}}{1+2x} = \frac{x^{1/2}}{1+2x}$$

$$F'(x)=\frac{(1+2x)(\frac{1}{2}x^{-1/2})-(x^{1/2})(2)}{(1+2x)^2}$$

$$=\frac{\frac{1}{2}x^{-1/2}+x^{1/2}-2x^{1/2}}{(1+2x)^2}$$

$$=\frac{\frac{1}{2}x^{-1/2}-x^{1/2}}{(1+2x)^2}$$

$$=\frac{\frac{1}{2\sqrt{x}}-\sqrt{x}}{(1+2x)^2}\quad 2\sqrt{x}\quad 2\sqrt{x}\quad 2\sqrt{x}$$

$$\boxed{=\frac{1-2x}{2\sqrt{x}(1+2x)^2}}$$

Differentiate the following functions, do not simplify your answers:

$$f(x)=\frac{8-9x^7}{3x-7}$$

$$F'(x)=\frac{(3x-7)(-63x^6)-(8-9x^7)(3)}{(3x-7)^2}$$

$$f(x)=\frac{x^3-7x^2+2}{x^8-4x^5}$$

$$F'(x)=\frac{(x^8-4x^5)(3x^2-14x)-(x^3-7x^2+2)(8x^7-20x^4)}{(x^8-4x^5)^2}$$

Homework

Ex 2.5

③ c) $y = \frac{1}{x^2+1}$; $(-2, \frac{1}{5})$

① $y' = \frac{(x^2+1)(0) - (1)(2x)}{(x^2+1)^2} = \boxed{\frac{-2x}{(x^2+1)^2}}$

② $y'(-2) = \frac{-2(-2)}{((-2)^2+1)^2} = \boxed{\frac{4}{25}}$ → "m"

③ $y - y_1 = m(x - x_1)$

$25 \cdot y - 25 \cdot \frac{1}{5} = 25 \cdot \frac{4}{25}(x+2)$

$25y - 5 = 4(x+2)$

$25y - 5 = 4x + 8$

$\boxed{0 = 4x - 25y + 13}$

Ex 2.5

4 $f(2)=3$ $\left(\frac{f}{g}\right)'(2) = ?$

$f'(2)=5$

$g(2)=-1$

$g'(2)=-4$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2}$$

$$= \frac{(-1)(5) - (3)(-4)}{(-1)^2}$$

$$= \frac{-5+12}{1}$$

$$\boxed{= 7}$$

Ex 2.5

(6) $y = \dfrac{x^2}{2x+5}$

$y' = \dfrac{(2x+5)(2x) - (x^2)(2)}{(2x+5)^2}$

$y' = \dfrac{4x^2 + 10x - 2x^2}{(2x+5)^2}$

$y' = \dfrac{2x^2 + 10x}{(2x+5)^2}$

$\dfrac{0}{1} = \dfrac{2x^2 + 10x}{(2x+5)^2}$

$2x^2 + 10x = 0$

$2x(x+5) = 0$

$2x = 0$	$x + 5 = 0$
$x = 0$	$x = -5$

$y = \dfrac{x^2}{2x+5}$

$y = \dfrac{(0)^2}{2(0)+5}$

$y = 0$

$y = \dfrac{(-5)^2}{2(-5)+5}$

$y = \dfrac{25}{-5}$

$y = -5$

$\boxed{(0,0) \text{ and } (-5,-5)}$

$$f(x) = \frac{3}{x^3} = 3x^{-3}$$

$$f'(x) = -9x^{-4} = \frac{-9}{x^4}$$

$$f(x) = \frac{3}{x^3}$$

$$f'(x) = \frac{(x^3)(0) - 3(3x^2)}{(x^3)^2}$$

$$= \frac{-9\cancel{x^2}}{x^{\cancel{6}}}$$

$$= \frac{-9}{x^4}$$

① b) $f(x) = \boxed{\dfrac{x+1}{x-1}}$ $\quad f(x+h) = \boxed{\dfrac{x+h+1}{x+h-1}}$

$$f'(x) = \lim_{h\to 0} \frac{\dfrac{x+h+1}{\cancel{x+h-1}}\,(x-1)\cancel{(x+h-1)} - \dfrac{x+1}{\cancel{x-1}}\,\cancel{(x-1)}(x+h-1)}{h\,(x-1)(x+h-1)}$$

$$= \lim_{h\to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h\to 0} \frac{\cancel{x^2} \cancel{+xh} \cancel{+x} \cancel{-x} - h \cancel{-1} - (\cancel{x^2} \cancel{+xh} \cancel{-x} \cancel{+x} + h \cancel{-1})}{h(x-1)(x+h-1)}$$

$$= \lim_{h\to 0} \frac{-2\cancel{h}}{\cancel{h}(x-1)(x+h-1)} \boxed{= \frac{-2}{(x-1)^2}}$$

5. $y = \frac{x+2}{3x+4}$

$y' = \frac{(3x+4)(1) - (x+2)(3)}{(3x+4)^2}$

$y' = \frac{3x+4-3x-6}{(3x+4)^2}$

$\boxed{y' = \frac{-2}{(3x+4)^2}}$ ← This is always negative

6. $y = \frac{x^2}{2x+5}$

Horizontal Lines have a slope of zero

$y' = \frac{(2x+5)(2x) - (x^2)(2)}{(2x+5)^2}$

$y' = \frac{4x^2+10x-2x^2}{(2x+5)^2}$

$y' = \frac{2x^2+10x}{(2x+5)^2}$

$\frac{0}{1} = \frac{2x^2+10x}{(2x+5)^2}$

$2x^2+10x = 0$

Factor
↳ $2x(x+5) = 0$

$2x=0$	$x+5=0$
$x=0$	$x=-5$
$y = \frac{x^2}{2x+5}$	$y = \frac{x^2}{2x+5}$
$y = \frac{(0)^2}{2(0)+5}$	$y = \frac{(-5)^2}{2(-5)+5}$
$y = 0$	$y = \frac{25}{-5}$
$(0,0)$	$y = -5$
	$(-5,-5)$

③ c) $y = \dfrac{1}{x^2+1}$; $\left(\overset{x_1}{-2}, \overset{y_1}{\tfrac{1}{5}}\right)$

(i) $y' = \dfrac{(x^2+1)(0) - 1(2x)}{(x^2+1)^2}$

$y' = \dfrac{-2x}{(x^2+1)^2}$

(ii) $y'(-2) = \dfrac{-2(-2)}{((-2)^2+1)^2}$

$y'(-2) = \dfrac{4}{25} = m.$

(iii) $y - y_1 = m(x - x_1)$

$y - \frac{1}{5} = \frac{4}{25}(x - (-2))$

$y - \frac{1}{5} = \frac{4}{25}(x+2)$

$y - \frac{1}{5} = \frac{4x}{25} + \frac{8}{25}$

$25y - 5 = 4x + 8$

$\boxed{0 = 4x - 25y + 13}$

④ $f(2)=\underline{\underline{3}}$

$f'(2)=\underline{\underline{5}}$

$g(2)=\underline{\underline{-1}}$

$g'(2)=\underline{\underline{-4}}$

$$\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x) - f(x)g'(x)}{[g(x)]^2}$$

$$\left(\frac{f}{g}\right)'(2) = \frac{g(2)f'(2) - f(2)g'(2)}{[g(2)]^2}$$

$$= \frac{(-1)(5) - (3)(-4)}{(-1)^2}$$

$$= \frac{-5+12}{1}$$

$$\boxed{=7}$$

Differentiation Quiz

① Limit Definition of the derivative

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

② Power Rule: $f(x) = x^n$ $\quad f'(x) = nx^{n-1}$

③ Product Rule: $(fg)' = fg' + f'g$

④ Quotient Rule: $\left(\frac{f}{g}\right)' = \frac{gf' - fg'}{g^2}$

Ex 2.5

⑤ $y = \frac{x+2}{3x+4}$

$y' = \frac{(3x+4)(1) - 3(x+2)}{(3x+4)^2}$

$y' = \frac{3x+4-3x-6}{(3x+4)^2}$

$y' = \frac{-2}{(3x+4)^2}$ ← always negative 2 ← always positive

⑥ $y = \frac{x^2}{2x+5}$

$y' = \frac{2x(2x+5) - 2x^2}{(2x+5)^2}$

$y' = \frac{4x^2+10x-2x^2}{(2x+5)^2}$

$y' = \frac{2x^2+10x}{(2x+5)^2}$

horizontal slope

$\frac{0}{1} = \frac{2x^2+10x}{(2x+5)^2}$

$2x^2+10x = 0$

$(2x)(x+5) = 0$

$2x = 0 \quad | \quad x+5=0$

$x = 0 \quad | \quad x = -5$

when $x=0$

$y = \frac{(0)^2}{2(0)+5} = 0$ $\boxed{(0,0)}$

when $x=-5$

$y = \frac{(-5)^2}{2(-5)+5} = -5$ $\boxed{(-5,-5)}$

⑦ $y = \frac{x}{x-1}$

$y' = \frac{(1)(x-1) - 1(x)}{(x-1)^2}$

$y' = \frac{-1}{(x-1)^2}$

$-\frac{1}{4} = \frac{-1}{(x-1)^2}$

$-(x-1)^2 = -4$

$(x-1)^2 = 4$

$x-1 = \pm 2$

$x-1=2 \quad | \quad x-1=-2$

$x=3 \quad | \quad x=-1$

parallel has same slope

$x+4y=1$

$4y = -x+1$

$y = -\frac{1}{4}x + \frac{1}{4}$

$m = -\frac{1}{4}$

$x=3$

$y = \frac{x}{x-1}$

$y = \frac{3}{2}$ $\boxed{(3, 3/2)}$

$x=-1$

$y = \frac{1}{2}$ $\boxed{(-1, 1/2)}$

① b) $f(x) = \dfrac{x+1}{x-1}$ $\quad f(x+h) = \dfrac{x+h+1}{x+h-1}$

$$f'(x) = \lim_{h \to 0} \frac{\frac{x+h+1}{x+h-1} - \frac{x+1}{x-1}}{h}$$

CD: $(x-1)(x+h-1)$

$$= \lim_{h \to 0} \frac{(x-1)(x+h+1) - (x+1)(x+h-1)}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{\cancel{x^2} \cancel{+xh} \cancel{+x} \cancel{-x} -h \cancel{-1} - (\cancel{x^2} \cancel{+xh} \cancel{-x} \cancel{+x} +h \cancel{-1})}{h(x-1)(x+h-1)}$$

$$= \lim_{h \to 0} \frac{-2\cancel{h}}{\cancel{h}(x-1)(x+h-1)} = \boxed{\frac{-2}{(x-1)^2}}$$

③ $y = x^3 + 3x$ at $(x=1)$

① Find y

$y = (1)^3 + 3(1)$

$y = 4$

Point $(1,4)$

② Find y'

$y' = 3x^2 + 3$

③ Find $y'(1)$

$y'(1) = 3(1)^2 + 3$

$y'(1) = 6$

$m = 6$

④ Find Equation:

$y - y_1 = m(x - x_1)$

$y - 4 = 6(x - 1)$

$y - 4 = 6x - 6$

$\boxed{0 = 6x - y - 2}$