

# Inverse of a Relation

## Focus on...

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- sketching the graph of the inverse of a relation
- determining if a relation and its inverse are functions
- determining the equation of an inverse

An inverse is often thought of as “undoing” or “reversing” a position, order, or effect. Whenever you undo something that you or someone else did, you are using an inverse, whether it is unwrapping a gift that someone else wrapped or closing a door that has just been opened, or deciphering a secret code.

## Inverse of a Relation

The inverse of a relation is found by interchanging the x-coordinates and y-coordinates of the ordered pairs of the relation. In other words, for every ordered pair  $(x, y)$  of a relation, there is an ordered pair  $(y, x)$  on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line  $y = x$ .

$$(x, y) \rightarrow (y, x)$$

The inverse of a function  $y = f(x)$  may be written in the form  $x = f(y)$ . The inverse of a function is not necessarily a function. When the inverse of  $f$  is itself a function, it is denoted as  $f^{-1}$  and read as “ $f$  inverse.” When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

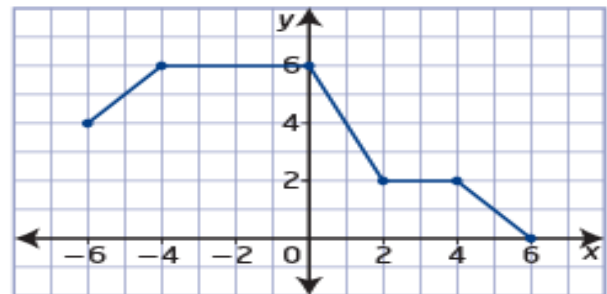
The inverse of a function reverses the processes represented by that function. Functions  $f(x)$  and  $g(x)$  are inverses of each other if the operations of  $f(x)$  reverse all the operations of  $g(x)$  in the opposite order and the operations of  $g(x)$  reverse all the operations of  $f(x)$  in the opposite order.

For example,  $f(x) = 2x + 1$  multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is  $f^{-1}(x) = \frac{x - 1}{2}$ .

### Example 1

#### Graph an Inverse

Consider the graph of the relation shown.



- Sketch the graph of the inverse relation.
- State the domain and range of the relation and its inverse.
- Determine whether the relation and its inverse are functions.

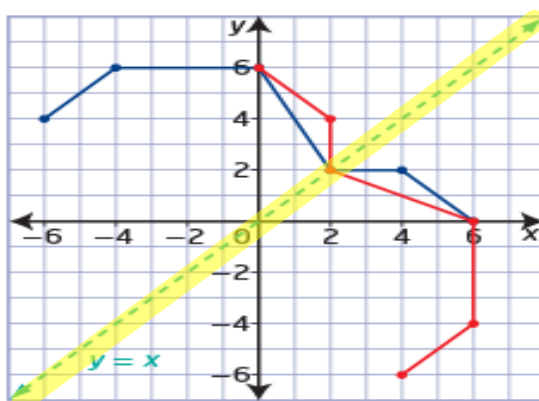
#### Solution

- To graph the inverse relation, interchange the  $x$ -coordinates and  $y$ -coordinates of key points on the graph of the relation.

Points on the Relation	Points on the Inverse Relation
$(-6, 4)$	$(4, -6)$
$(-4, 6)$	$(6, -4)$
$(0, 6)$	$(6, 0)$
$(2, 2)$	$(2, 2)$
$(4, 2)$	$(2, 4)$
$(6, 0)$	$(0, 6)$

Blue curve

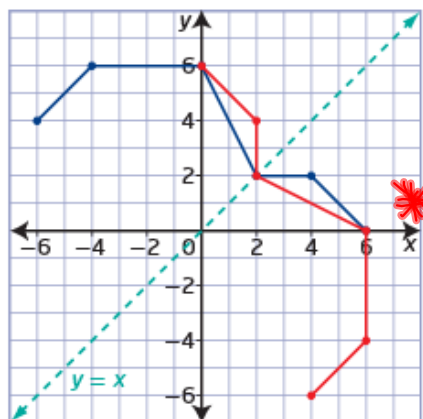
Red Curve



The graphs are reflections of each other in the line  $y = x$ . The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping  $(x, y) \rightarrow (y, x)$ .

What points are invariant after a reflection in the line  $y = x$ ?

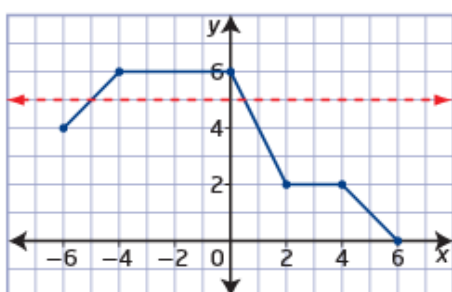
b) State the domain and range of the relation and its inverse.



	Domain	Range
Relation	$\{x \mid -6 \leq x \leq 6, x \in \mathbb{R}\}$	$\{y \mid 0 \leq y \leq 6, y \in \mathbb{R}\}$
Inverse Relation	$\{x \mid 0 \leq x \leq 6, x \in \mathbb{R}\}$	$\{y \mid -6 \leq y \leq 6, y \in \mathbb{R}\}$

The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

c) Determine whether the relation and its inverse are functions.



#### horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

The inverse relation is not a function of  $x$  because it fails the vertical line test. There is more than one value of  $y$  in the range for at least one value of  $x$  in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.

## Inverse of a Relation

### Key Ideas

- You can find the inverse of a relation by interchanging the  $x$ -coordinates and  $y$ -coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line  $y = x$ .
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function  $f(x)$  is itself a function, it is denoted by  $f^{-1}(x)$ .
- You can verify graphically whether two functions are inverses of each other.

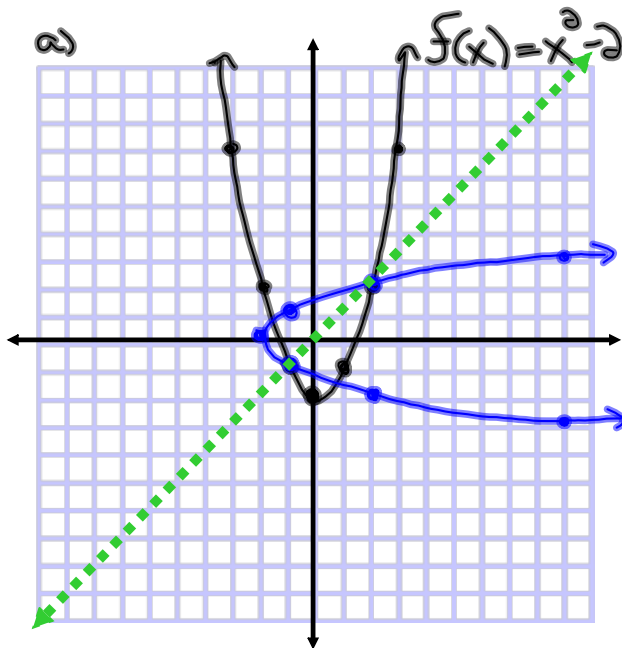
### Example 2

#### Restrict the Domain

Consider the function  $f(x) = x^2 - 2$ .

*Use horizontal line test*

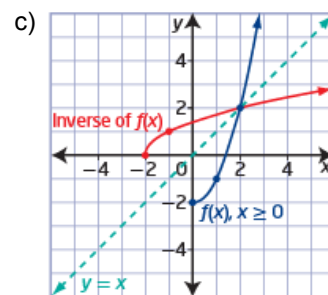
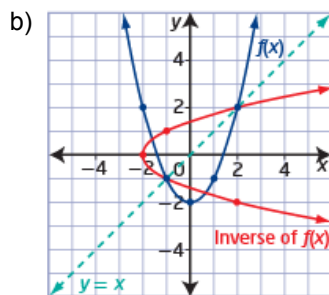
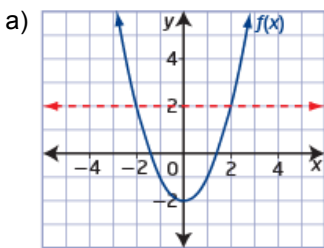
- Graph the function  $f(x)$ . Is the inverse of  $f(x)$  a function?  $\rightarrow$  No
- Graph the inverse of  $f(x)$  on the same set of coordinate axes.
- Describe how the domain of  $f(x)$  could be restricted so that the inverse of  $f(x)$  is a function.



$(x, y) \rightarrow (y, x)$

$f(x) = x^2 - 2$		Inverse	
-2	2	2	-2
-1	1	1	-1
0	-2	-2	0
1	1	1	-1
2	2	2	-2

#### Solutions



Ex:

$$f(x) = (x+3)^2$$

$$x+3 \geq 0$$

$$x \geq -3$$

$$D: \{x \mid x \geq -3, x \in \mathbb{R}\}$$

- c) The inverse of  $f(x)$  is a function if the graph of  $f(x)$  passes the horizontal line test.

One possibility is to restrict the domain of  $f(x)$  so that the resulting graph is only one half of the parabola.

Since the equation of the axis of symmetry is  $x = 0$ , restrict the domain to  $\{x \mid x \geq 0, x \in \mathbb{R}\}$ .

### Example 3

#### Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function.

Verify graphically that the relations are inverses of each other.

a)  $f(x) = 3x + 6$  ← Linear

b)  $f(x) = x^2 - 4$  ← Quadratic

a)  $f(x) = 3x + 6$

①  $y = 3x + 6$

②  $x = 3y + 6$

③  $x - 6 = 3y$

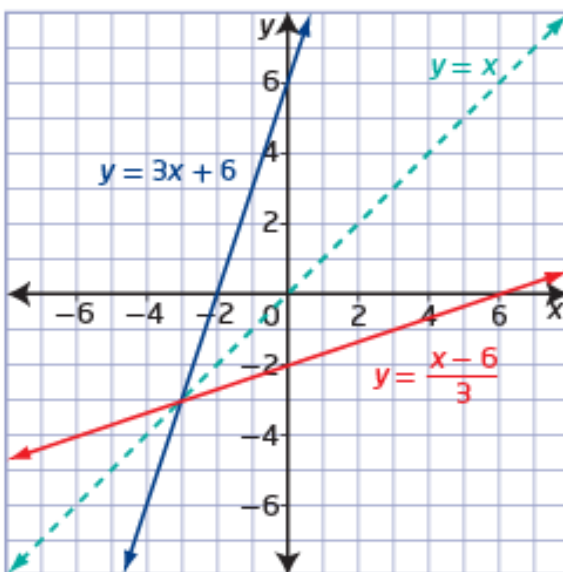
$$\frac{x-6}{3} = y$$

$$y = \frac{x-6}{3} = \frac{1}{3}x - 2$$

④  $f^{-1}(x) = \frac{x-6}{3}$

- 1) Replace  $f(x)$  with  $y$ .
- 2) Switch  $x$ 's and  $y$ 's.
- 3) Solve for  $y$ .
- 4) Replace  $y$  with  $f^{-1}(x)$ .  
(if the inverse is a function!)

Graph  $y = 3x + 6$  and  $y = \frac{x-6}{3}$  on the same set of coordinate axes.





### Determine the Equation of the Inverse

b)  $f(x) = x^2 - 4$  (Quadratic)

$$y = x^2 - 4$$

$$x = y^2 - 4$$

$$x + 4 = y^2$$

$$\pm\sqrt{x+4} = y$$

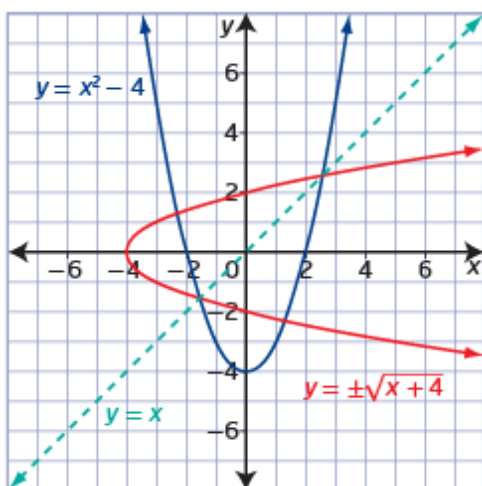
Why is this  $y$  not replaced with  $f^{-1}(x)$ ? What could be done so that  $f^{-1}(x)$  could be used?

(Not a function)

restrict domain of  $f(x) \rightarrow x+4 \geq 0$   
 $x \geq -4$

- 1) Replace  $f(x)$  with  $y$ .
- 2) Switch  $x$ 's and  $y$ 's.
- 3) Solve for  $y$ .
- 4) Replace  $y$  with  $f^{-1}(x)$ .  
 (if the inverse is a function!)

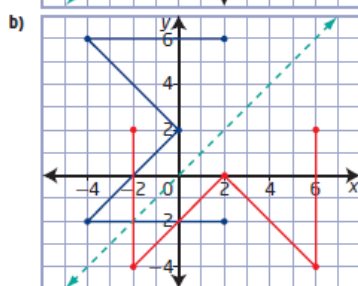
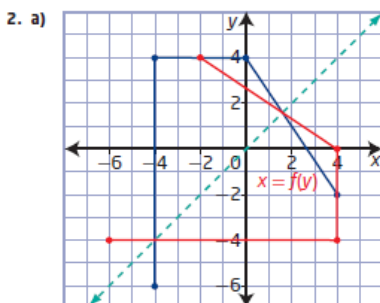
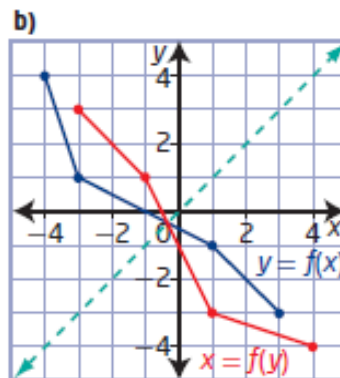
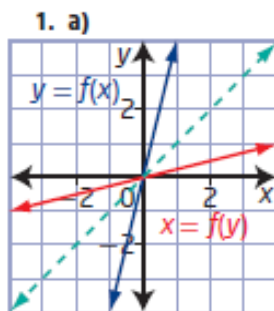
Graph  $y = x^2 - 4$  and  $y = \pm\sqrt{x+4}$  on the same set of coordinate axes.



## Homework

finish #1-11 on page 51

1.4 Inverse of a Relation, pages 51 to 55

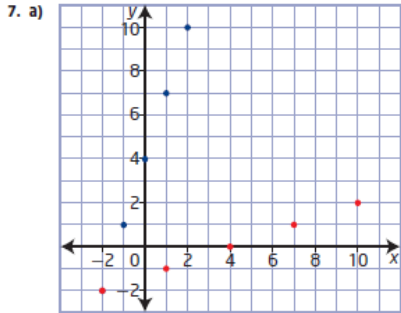


3. a) The graph is a function but the inverse will be a relation.  
 b) The graph and its inverse are functions.  
 c) The graph and its inverse are relations.

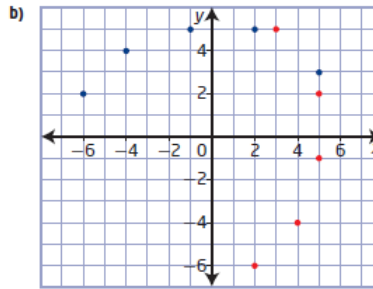
4. Examples:

- a)  $\{x \mid x \geq 0, x \in \mathbb{R}\}$  or  $\{x \mid x \leq 0, x \in \mathbb{R}\}$   
 b)  $\{x \mid x \geq -2, x \in \mathbb{R}\}$  or  $\{x \mid x \leq -2, x \in \mathbb{R}\}$   
 c)  $\{x \mid x \geq 4, x \in \mathbb{R}\}$  or  $\{x \mid x \leq 4, x \in \mathbb{R}\}$   
 d)  $\{x \mid x \geq -4, x \in \mathbb{R}\}$  or  $\{x \mid x \leq -4, x \in \mathbb{R}\}$   
 5. a)  $f^{-1}(x) = \frac{1}{7}x$       b)  $f^{-1}(x) = -\frac{1}{3}(x - 4)$   
 c)  $f^{-1}(x) = 3x - 4$       d)  $f^{-1}(x) = 3x + 15$   
 e)  $f^{-1}(x) = -\frac{1}{2}(x - 5)$       f)  $f^{-1}(x) = 2x - 6$

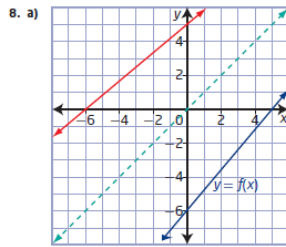
6. a) E    b) C    c) B    d) A    e) D



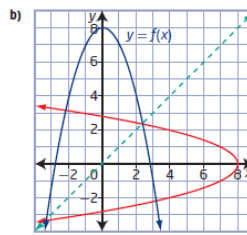
function: domain  $\{-2, -1, 0, 1, 2\}$ ,  
range  $\{-2, 1, 4, 7, 10\}$   
inverse: domain  $\{-2, 1, 4, 7, 10\}$ ,  
range  $\{-2, -1, 0, 1, 2\}$



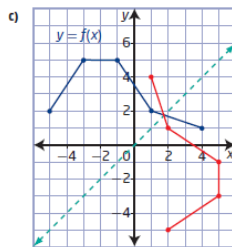
function: domain  $\{-6, -4, -1, 2, 5\}$ , range  $\{2, 3, 4, 5\}$   
inverse: domain  $\{2, 3, 4, 5\}$ , range  $\{-6, -4, -1, 2, 5\}$



The inverse is a function; it passes the vertical line test.

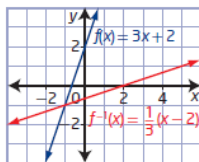


The inverse is not a function; it does not pass the vertical line test.



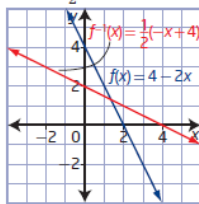
The inverse is not a function; it does not pass the vertical line test.

9. a)  $f^{-1}(x) = \frac{1}{3}(x - 2)$



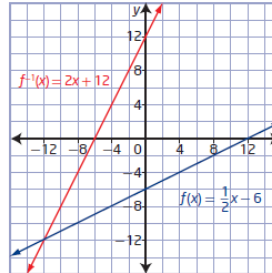
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$

b)  $f^{-1}(x) = \frac{1}{2}(-x + 4)$



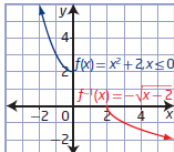
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$

c)  $f^{-1}(x) = 2x + 12$



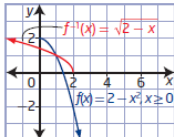
$f(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \in \mathbb{R}\}$ , range  $\{y \mid y \in \mathbb{R}\}$

d)  $f^{-1}(x) = -\sqrt{x-2}$



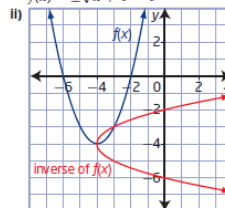
$f(x)$ : domain  $\{x \mid x \leq 0, x \in \mathbb{R}\}$ , range  $\{y \mid y \geq 2, y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \geq 2, x \in \mathbb{R}\}$ , range  $\{y \mid y \leq 0, y \in \mathbb{R}\}$

e)  $f^{-1}(x) = \sqrt{2-x}$

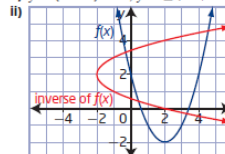


$f(x)$ : domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$ , range  $\{y \mid y \leq 2, y \in \mathbb{R}\}$   
 $f^{-1}(x)$ : domain  $\{x \mid x \leq 2, x \in \mathbb{R}\}$ , range  $\{y \mid y \geq 0, y \in \mathbb{R}\}$

10. a) i)  $f(x) = (x + 4)^2 - 4$ , inverse of  $f(x) = \pm\sqrt{x+4} - 4$

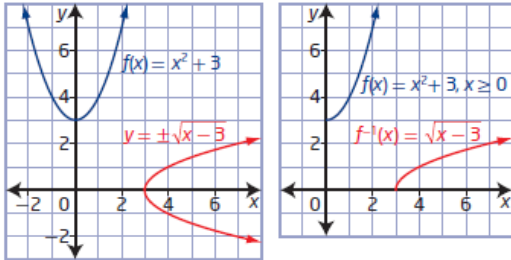


b) i)  $y = (x - 2)^2 - 2$ ,  $y = \pm\sqrt{x+2} + 2$

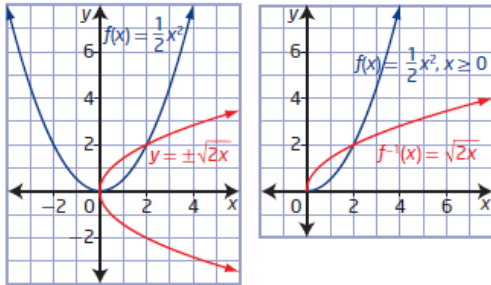


11. Yes, the graphs are reflections of each other in the line  $y = x$ .

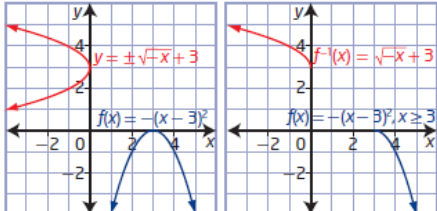
12. a)  $y = \pm\sqrt{x-3}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



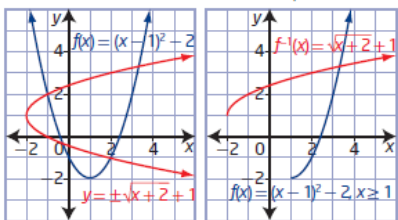
b)  $y = \pm\sqrt{2x}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



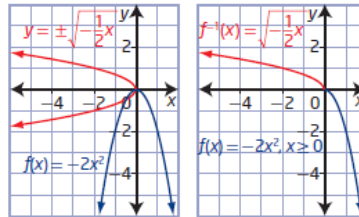
e)  $y = \pm\sqrt{-x+3}$  restricted domain  $\{x \mid x \geq 3, x \in \mathbb{R}\}$



f)  $y = \pm\sqrt{x+2} + 1$  restricted domain  $\{x \mid x \geq 1, x \in \mathbb{R}\}$



c)  $y = \pm\sqrt{-\frac{1}{2}x}$  restricted domain  $\{x \mid x \geq 0, x \in \mathbb{R}\}$



d)  $y = \pm\sqrt{x-1}$  restricted domain  $\{x \mid x \geq -1, x \in \mathbb{R}\}$

