Function Operations

To combine two functions, $f(x)$ and $g(x)$, multiply or divide as follows:

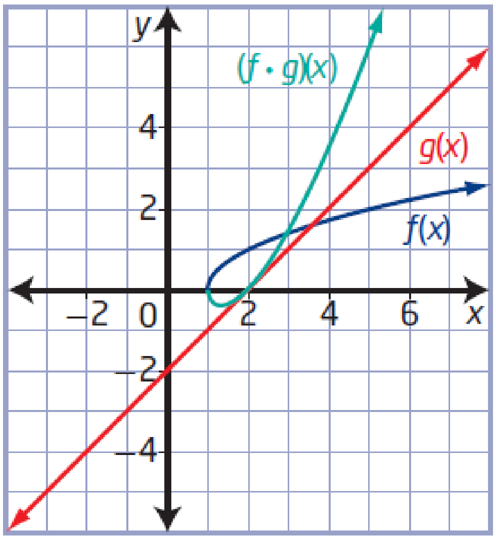
Product of Functions	*Quotient of Functions*
$h(x) = f(x)g(x)$	$h(x) = \frac{f(x)}{g(x)}$
$h(x) = (f \cdot g)(x)$	$h(x) = \left(\frac{f}{g}\right)(x)$

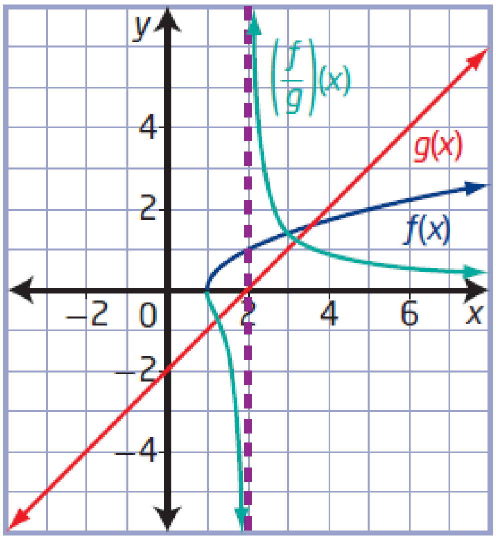
The domain of a product of functions is the domain common to the original functions. However, the domain of a quotient of functions must take into consideration that division by zero is undefined. The domain of a quotient, $h(x) = \frac{f(x)}{g(x)}$, is further restricted for values of x where $g(x) = 0$.

Example

Consider $f(x) = \sqrt{x - 1}$ and $g(x) = x - 2$.

The domain of $f(x)$ is $\{x \mid x \geq 1, x \in R\}$, and the domain of $g(x)$ is $\{x \mid x \in R\}$. So, the domain of $(f \cdot g)(x)$ is , while the domain of $\left(\frac{f}{g}\right)(x)$ is





Key Ideas

- The combined function $h(x) = (f \cdot g)(x)$ represents the product of two functions, $f(x)$ and $g(x)$.
- The combined function $h(x) = \left(\frac{f}{g}\right)(x)$ represents the quotient of two functions, $f(x)$ and $g(x)$, where $g(x) \neq 0$.
- The domain of a product or quotient of functions is the domain common to both $f(x)$ and $g(x)$. The domain of the quotient $\left(\frac{f}{g}\right)(x)$ is further restricted by excluding values where $g(x) = 0$.
- The range of a combined function can be determined using its graph.

Example 1

Determine the Product of Functions

multiply

Given $f(x) = (x + 2)^2 - 5$ and $g(x) = 3x - 4$, determine $h(x) = (f \cdot g)(x)$. State the domain and range of $h(x)$.

Solution

To determine $h(x) = (f \cdot g)(x)$, multiply the two functions.

$h(x) = (f \cdot g)(x)$
$h(x) = f(x)g(x)$
$h(x) = ((x + 2)^2 - 5)(3x - 4)$
$h(x) = (x^2 + 4x - 1)(3x - 4)$
$h(x) = 3x^3 - 4x^2 + 12x^2 - 16x - 3x + 4$
$h(x) = 3x^3 + 8x^2 - 19x + 4$

How can you tell from the original functions that the product is a cubic function?

The function $f(x) = (x + 2)^2 - 5$ is quadratic with domain $\{x \mid x \in R\}$.

The function $g(x) = 3x - 4$ is linear with domain $\{x \mid x \in R\}$.

The domain of $h(x) = (f \cdot g)(x)$ consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$.

Therefore, the cubic function $h(x) = 3x^3 + 8x^2 - 19x + 4$ has domain $\{x \mid x \in R\}$ and range $\{y \mid y \in R\}$.

Example 2

Determine the Quotient of Functions

Consider the functions $f(x) = x^2 + x - 6$ and $g(x) = 2x + 6$.

a) Determine the equation of the function $h(x) = \left(\frac{g}{f}\right)(x)$. ← Divide

b) Sketch the graphs of $f(x)$, $g(x)$, and $h(x)$ on the same set of coordinate axes.

c) State the domain and range of $h(x)$.

Solution

a) To determine $h(x) = \left(\frac{g}{f}\right)(x)$, divide the two functions.

$$h(x) = \left(\frac{g}{f}\right)(x)$$

$$h(x) = \frac{g(x)}{f(x)}$$

common factor of 2

$$h(x) = \frac{2x + 6}{x^2 + x - 6}$$

simple trinomial $3 \times -2 = -6$, $3 + -2 = 1$

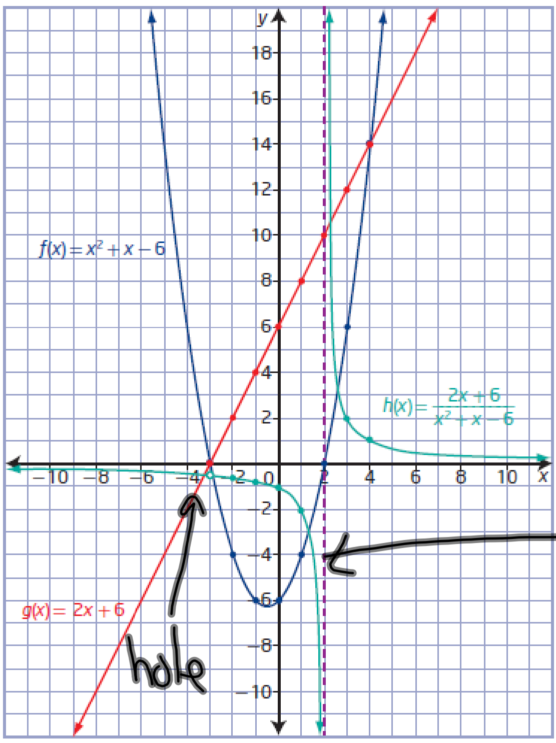
$$h(x) = \frac{2(x + 3)}{(x + 3)(x - 2)} \qquad \text{Factor.}$$

$$h(x) = \frac{2\overset{1}{\cancel{(x + 3)}}}{\underset{1}{\cancel{(x + 3)}}(x - 2)}$$

$$h(x) = \frac{2}{x - 2}, x \neq -3, 2 \qquad \text{Identify any non-permissible values.}$$

b) **Method 1: Use Paper and Pencil**

x	$f(x) = x^2 + x - 6$	$g(x) = 2x + 6$	$h(x) = \frac{2}{x - 2}, x \neq -3, 2$
-3	0	0	does not exist
-2	-4	2	$-\frac{1}{2}$
-1	-6	4	$-\frac{2}{3}$
0	-6	6	-1
1	-4	8	-2
2	0	10	undefined
3	6	12	2
4	14	14	1



How are the y-coordinates of points on the graph of $h(x)$ related to those on the graphs of $f(x)$ and $g(x)$?

vertical asymptote

c) The function $f(x) = x^2 + x - 6$ is quadratic with domain $\{x \mid x \in \mathrm{R}\}$.
The function $g(x) = 2x + 6$ is linear with domain $\{x \mid x \in \mathrm{R}\}$.
The domain of $h(x) = \left(\frac{g}{f}\right)(x)$ consists of all values that are in both the domain of $f(x)$ and the domain of $g(x)$, excluding values of x where $f(x) = 0$.

Since the function $h(x)$ does not exist at $\left(-3, -\frac{2}{5}\right)$ and is undefined at $x = 2$, the domain is $\{x \mid x \neq -3, x \neq 2, x \in \mathrm{R}\}$. This is shown in the graph by the point of discontinuity at $\left(-3, -\frac{2}{5}\right)$ and the vertical asymptote that appears at $x = 2$.

$f(x) = \frac{2}{x-2}$

$f(-3) = \frac{2}{-3-2}$

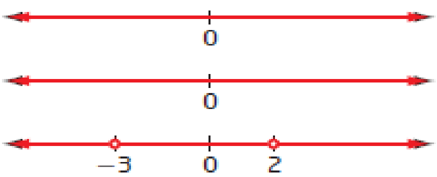
$= -\frac{2}{5}$

How do you know there is a point of discontinuity and an asymptote?

Domain of $f(x) = x^2 + x - 6$

Domain of $g(x) = 2x + 6$

Domain of $h(x) = \left(\frac{g}{f}\right)(x)$



The range of $h(x)$ is $\left\{y \mid y \neq 0, -\frac{2}{5}, y \in \mathrm{R}\right\}$.

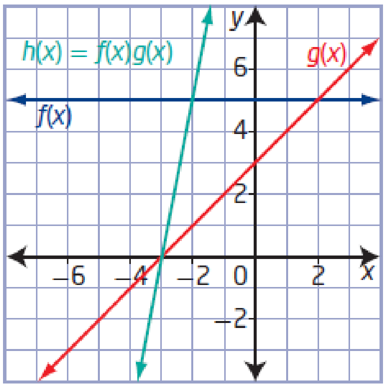
Homework

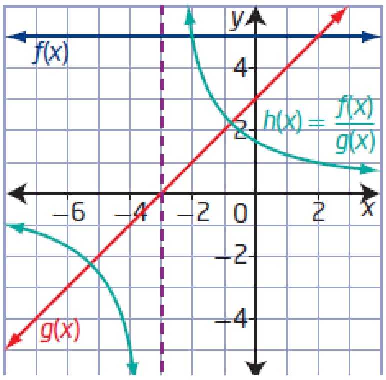
finish #1-9 on page 496-497

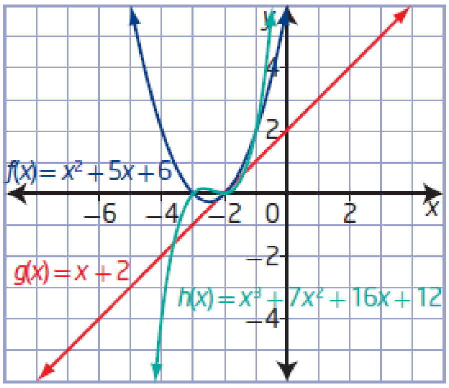
10.2 Products and Quotients of Functions, pages 496 to 498

1. a) $h(x) = x^2 - 49$, $k(x) = \dfrac{x + 7}{x - 7}$, $x \neq 7$
   b) $h(x) = 6x^2 + 5x - 4$, $k(x) = \dfrac{2x - 1}{3x + 4}$, $x \neq -\dfrac{4}{3}$
   c) $h(x) = (x + 2)\sqrt{x + 5}$, $k(x) = \dfrac{\sqrt{x + 5}}{x + 2}$, $x \geq -5$, $x \neq -2$
   d) $h(x) = \sqrt{-x^2 + 7x - 6}$, $k(x) = \dfrac{\sqrt{x - 1}}{\sqrt{6 - x}}$, $1 \leq x < 6$

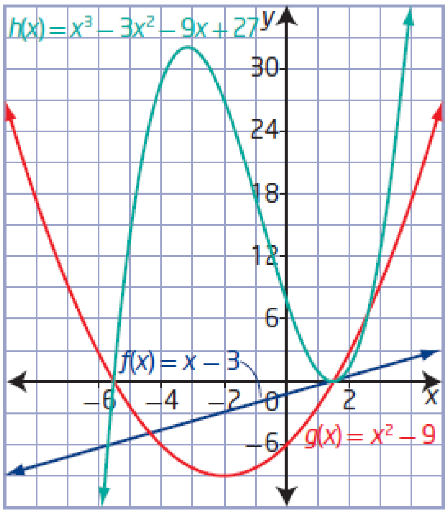
2. a) -3 b) 0 c) -1 d) 0

3. a)


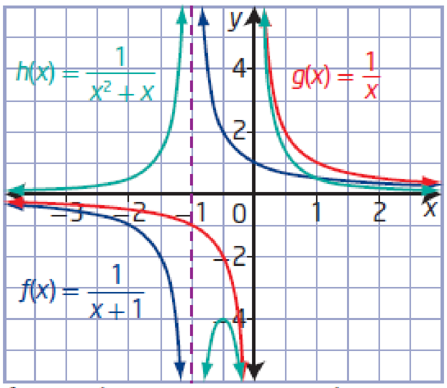
b)


4. a) $h(x) = x^3 + 7x^2 + 16x + 12$


domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

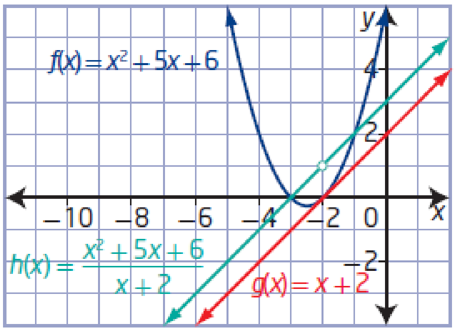
b) $h(x) = x^3 - 3x^2 - 9x + 27$


domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

c) $h(x) = \dfrac{1}{x^2 + x}$


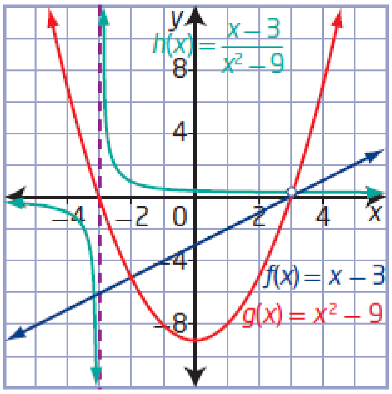
domain $\{x \mid x \neq 0, -1, x \in \mathrm{R}\}$,
range $\{y \mid y \leq -4 \text{ or } y > 0, y \in \mathrm{R}\}$

5. a) $h(x) = x + 3,\ x \neq -2$



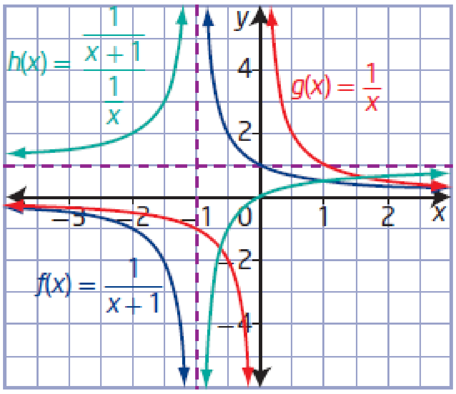
domain $\{x \mid x \neq -2, x \in \mathrm{R}\}$, range $\{y \mid y \neq 1, y \in \mathrm{R}\}$

b) $h(x) = \dfrac{1}{x + 3},\ x \neq \pm 3$



domain $\{x \mid x \neq \pm 3, x \in \mathrm{R}\}$, range $\left\{y \mid y \neq 0, \frac{1}{6}, y \in \mathrm{R}\right\}$

c) $h(x) = \dfrac{x}{x + 1},\ x \neq -1, 0$



domain $\{x \mid x \neq -1, 0, x \in \mathrm{R}\}$, range $\{y \mid y \neq 0, 1, y \in \mathrm{R}\}$

6. a) $y = x^3 + 3x^2 - 10x - 24$

b) $y = \dfrac{x^2 - x - 6}{x + 4},\ x \neq -4$ **c)** $y = \dfrac{2x - 1}{x + 4},\ x \neq -4$

d) $y = \dfrac{x^2 - x - 6}{x^2 + 8x + 16},\ x \neq -4$

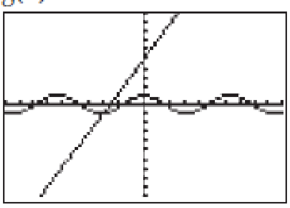
7. a) $g(x) = 3$ **b)** $g(x) = -x$

c) $g(x) = \sqrt{x}$ **d)** $g(x) = 5x - 6$

8. a) $g(x) = x + 7$ **b)** $g(x) = \sqrt{x + 6}$

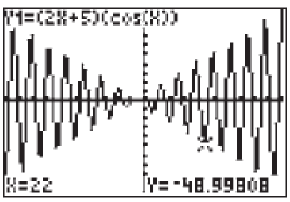
c) $g(x) = 2$ **d)** $g(x) = 3x^2 + 26x - 9$

9. a)



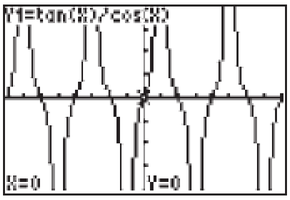
$f(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$
$g(x)$: domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid -1 \leq y \leq 1, y \in \mathrm{R}\}$

b)



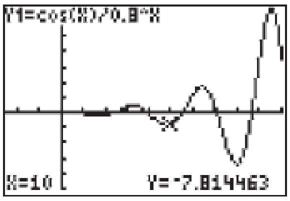
domain $\{x \mid x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$

10. a)



domain $\{x \mid x \neq (2n - 1)\frac{\pi}{2}, n \in I, x \in R\}$, range $\{y \mid y \in R\}$

b)



domain $\{x \mid x \in R\}$, range $\{y \mid y \in R\}$

11. a) $y = \frac{f(x)}{g(x)}$ **b)** $y = f(x)f(x)$

c) The graphs of $y = \frac{\sin x}{\cos x}$ and $y = \tan x$ appear to be the same. The graphs of $y = 1 - \cos^2 x$ and $y = \sin^2 x$ appear to be the same.