

Warm Up

Differentiate the following:

$$f(x) = \frac{-2x \tan^{-1} \sqrt{x}}{\cos^{-1}(\sec x^3)}$$

$$d \tan^{-1} u = \frac{1}{1+u^2} \cdot du = \frac{du}{1+u^2}$$

$$d \cos^{-1} u = \frac{-1}{\sqrt{1-u^2}} \cdot du = \frac{-du}{\sqrt{1-u^2}}$$

$$d \sec u = \sec u \tan u \cdot du$$

$$f'(x) = \cos^{-1}(\sec x^3) \left[-2x \left(\frac{1}{1+x} \cdot \frac{1}{2} x^{-1/2} \right) + \tan^{-1} \sqrt{x} (-2) \right] -$$
$$(-2x \tan^{-1} \sqrt{x}) \left[\frac{-1}{\sqrt{1-\sec^2 x^3}} \cdot \sec x^3 \tan x^3 \cdot 3x^2 \right]$$

$$\left[\cos^{-1}(\sec x^3) \right]^2$$

Questions from Homework

$$\textcircled{2} f(x) = x \tan^{-1} x$$

$$f'(x) = x \left[\frac{1}{1+x^2} \cdot 1 \right] + 1(\tan^{-1} x)$$

$$f'(x) = \frac{x}{1+x^2} + \tan^{-1} x$$

$$f'(1) = \frac{1}{1+(1)^2} + \tan^{-1}(1)$$

← what angle has a tangent value equal to 1

$$= \frac{1}{2} + \frac{\pi}{4}$$

$$= \frac{2 + \pi}{4}$$

$$\textcircled{4} f(x) = (3 \tan^{-1} x)^4$$

$$f'(x) = 4(3 \tan^{-1} x)^3 \left[3 \left(\frac{1}{1+x^2} \cdot 1 \right) + \cancel{(2) \tan^{-1} x} \right]$$

$$f'(x) = 4(3 \tan^{-1} x)^3 \left[\frac{3}{1+x^2} \right]$$

$$f'(x) = \frac{12(3 \tan^{-1} x)^3}{1+x^2}$$

$$f'(\sqrt{3}) = \frac{12(3 \tan^{-1} \sqrt{3})^3}{1+(\sqrt{3})^2}$$

← what angle has a tangent value equal to $\sqrt{3}$

$$= \frac{12(3(\frac{\pi}{3}))^3}{1+3}$$

$$= \frac{12\pi^3}{4}$$

$$= 3\pi^3$$

$$\textcircled{6} f(x) = (x-3)(6x-x^2)^{1/2} + 9 \sin^{-1} \left(\frac{x-3}{3} \right)$$

$$f'(x) = (x-3) \frac{1}{2} (6x-x^2)^{-1/2} (6-2x) + (6x-x^2)^{1/2} + 9 \left[\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}} \cdot \frac{1}{3} \right]$$

$$f'(x) = \frac{(x-3)(6-2x)}{2\sqrt{6x-x^2}} + \sqrt{6x-x^2} + \frac{3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}$$

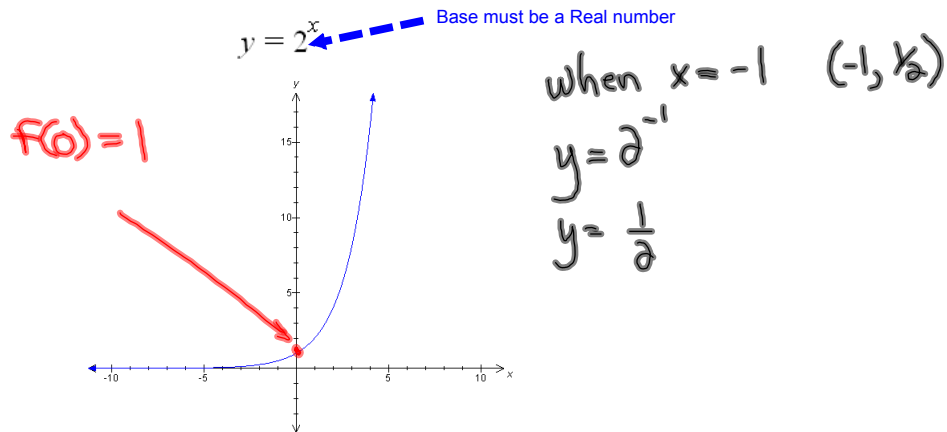
$$f'(3) = \frac{(0)(0)}{6} + 3 + 3$$

$$= 0 + 3 + 3$$

$$= 6$$

Differentiating Exponential Functions

What is an exponential function? $y = a^x$



When you do not have a rule to differentiate resort to the definition...

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

Let's try and differentiate $y = a^x$ $f(x) = a^x$ $f(x+h) = a^{x+h}$

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} \\ &= \lim_{h \rightarrow 0} \frac{a^x a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x (a^h - 1)}{h} \end{aligned}$$

This factor does not depend on h , therefore we can move to the front of the limit

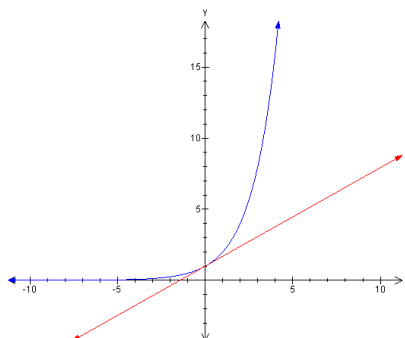
Thus we now have...

$$f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

What would be the value of $f'(0)$?

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



We have determined that $f'(x) = a^x \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$

and that $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = f'(0)$

Same thing!

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

■ $a=2$; here apparently $f'(0) \approx 0.69$
 ■ $a=3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "e"...Euler's Number

`e^(1)`
2.718281828

$y = 2^x$
 $y' = 2^x (0.6932)$

$y = e^x$
 $y' = e^x(1)$
 $y' = e^x$

This leads to the following definition...

Definition of the Number e

$$e \text{ is the number such that } \lim_{h \rightarrow 0} \frac{e^h - 1}{h} = 1$$

What does this mean geometrically?

- Geometrically, this means that
 - of all the exponential functions $y = a^x$,
 - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1.

$$f(0) = 1$$
$$f'(0) = 1$$

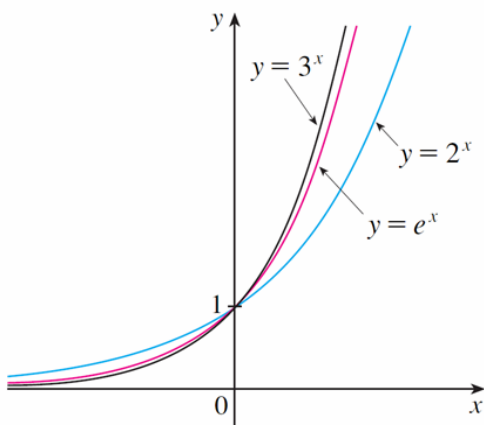


FIGURE 6

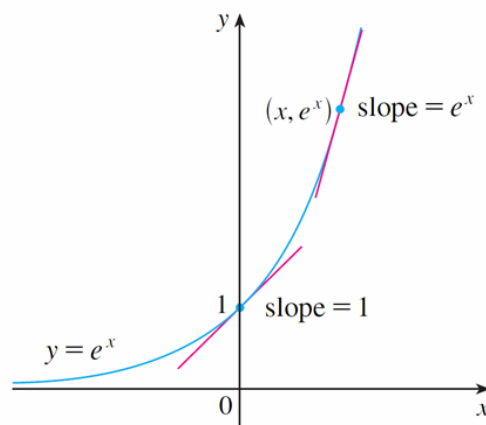


FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

**This is the ONLY function
that is its own derivative**

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$$\begin{aligned} f(x) &= e^x \\ f'(x) &= e^x \end{aligned}$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

Differentiating Exponential Functions

$$u = 3x^7 \quad du = 21x^6$$

$$y = e^{3x^7}$$

$$y' = e^{3x^7} \cdot 21x^6$$

$$y' = 21x^6 e^{3x^7}$$

$$u = \sin x \quad du = \cos x$$

$$y = e^{\sin x}$$

$$y' = e^{\sin x} \cdot \cos x$$

$$y = x^2 e^x$$

$$y' = x^2 e^x (1) + 2x e^x$$

$$y' = x e^x (x + 2)$$

$$u = \cot x^3 \quad du = -\csc^2 x^3 \cdot 3x^2$$

$$y = e^{\cot x^3}$$

$$y' = e^{\cot x^3} \cdot -3x^2 \csc^2 x^3$$

$$y = e^{-3x} \cos 2x$$

$$y' = e^{-3x} (-\sin 2x)(2) + e^{-3x} (-3)(\cos 2x)$$

$$y' = -e^{-3x} (2 \sin 2x) - e^{-3x} (3 \cos 2x)$$

$$y' = -e^{-3x} (2 \sin 2x + 3 \cos 2x)$$

Practice Exercises

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#4, 5, 6, 8, 9, 10,

⑤ $y = 1 + xe^{2x}$ at $x=0$ Point: $(0, 1)$
 $y = 1 + (0)e^{2(0)}$ (x, y)
 $y = 1 + 0$
 $y = 1$

① Differentiate:

$$y = 1 + xe^{2x}$$

$$y' = 0 + x(e^{2x} \cdot 2) + 1e^{2x}$$

$$y' = 2xe^{2x} + e^{2x}$$

$$y' = e^{2x}(2x+1)$$

② Sub in x-value:

$$y' = e^{2(0)}(2(0)+1)$$

$$y' = 1(1)$$

$$y' = 1 \leftarrow \text{"m"}$$

③ Find equation:

$$y - y_1 = m(x - x_1)$$

$$y - 1 = 1(x - 0)$$

$$y - 1 = x$$

$$\boxed{0 = x - y + 1}$$

⑥ $e^{xy} = 2x + y$ Implicit Diff.

$$e^{xy}(xy' + y) = 2 + y'$$

$$xy'e^{xy} + ye^{xy} = 2 + y'$$

$$xy'e^{xy} - y' = 2 - ye^{xy}$$

$$y'[xe^{xy} - 1] = 2 - ye^{xy}$$

$$\boxed{y' = \frac{2 - ye^{xy}}{xe^{xy} - 1}}$$

Bonus:

Give that $y = \cos^{-1}(\cos^{-1} x)$, prove that

$$\frac{dy}{dx} = \frac{1}{\sin y \sqrt{1-x^2}}$$