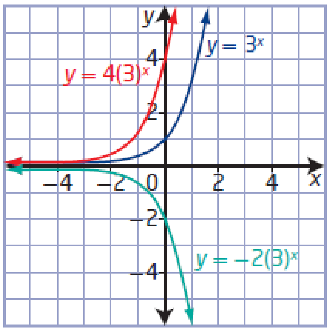
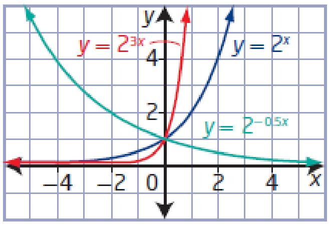
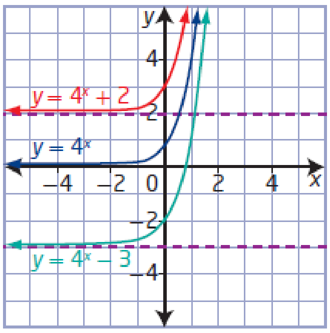
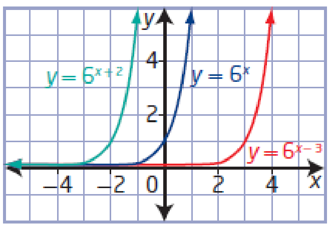
Transformations of Exponential Functions

Focus on...

- applying translations, stretches, and reflections to the graphs of exponential functions
- representing these transformations in the equations of exponential functions
- solving problems that involve exponential growth or decay

Link the Ideas

The graph of a function of the form $f(x) = a(c)^{b(x-h)} + k$ is obtained by applying transformations to the graph of the base function $y = c^x$, where $c > 0$.

Parameter	Transformation	Example
a	• Vertical stretch about the x-axis by a factor of $\lvert a \rvert$ • For $a < 0$, reflection in the x-axis • $(x, y) \rightarrow (x, ay)$	
b	• Horizontal stretch about the y-axis by a factor of $\frac{1}{\lvert b \rvert}$ • For $b < 0$, reflection in the y-axis • $(x, y) \rightarrow \left(\frac{x}{b}, y\right)$	
k	• Vertical translation up or down • $(x, y) \rightarrow (x, y + k)$	
h	• Horizontal translation left or right • $(x, y) \rightarrow (x + h, y)$	

Example 1

Apply Transformations to Sketch a Graph

Consider the base function $y = 3^x$. For each transformed function,

i) state the parameters and describe the corresponding transformations

ii) create a table to show what happens to the given points under each transformation

$y = 3^x$
$\left(-1, \frac{1}{3}\right)$
(0, 1)
(1, 3)
(2, 9)
(3, 27)

iii) sketch the graph of the base function and the transformed function

iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

a) $y = 2(3)^{x-4}$

Solution

a) **i)** Compare the function $y = 2(3)^{x-4}$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

- $b = 1$ corresponds to no horizontal stretch.
- $a = 2$ corresponds to a vertical stretch of factor 2. Multiply the y-coordinates of the points in column 1 by 2.
- $h = 4$ corresponds to a translation of 4 units to the right. Add 4 to the x-coordinates of the points in column 2.
- $k = 0$ corresponds to no

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 2(3)^x$	$y = 2(3)^{x-4}$
$\left(-1, \frac{1}{3}\right)$	$\left(-1, \frac{2}{3}\right)$	$\left(3, \frac{2}{3}\right)$
(0, 1)	(0, 2)	(4, 2)
(1, 3)	(1, 6)	(5, 6)
(2, 9)	(2, 18)	(6, 18)
(3, 27)	(3, 54)	(7, 54)

y int

$(x, y) \to (x+4, 2y)$

what is the y-int

Let $x=0$

$y = 2(3)^{0-4}$

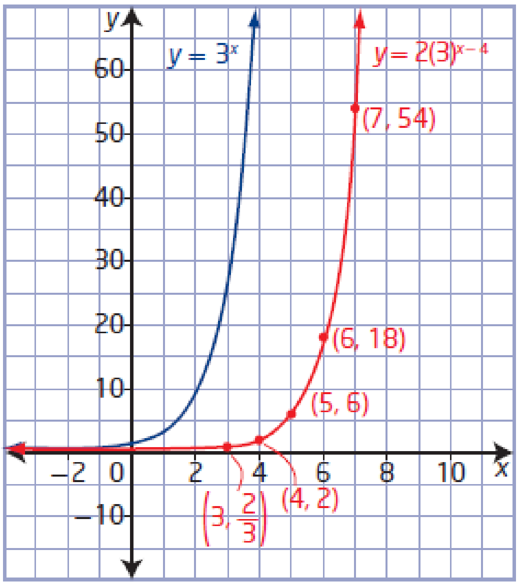
$y = 2(3)^{-4}$

$y = 2\left(\frac{1}{3}\right)^4$

$y = 2\left(\frac{1}{81}\right) = \frac{2}{81}$

$\left(0, \frac{2}{81}\right)$

iii) To sketch the graph, plot the points from column 3 and draw a smooth curve through them.



iv) The domain remains the same: $\{x \mid x \in \mathrm{R}\}$.

The range also remains unchanged: $\{y \mid y > 0, y \in \mathrm{R}\}$.

The equation of the asymptote remains as $y = 0$.

There is still no x-intercept, but the y-intercept changes to $\frac{2}{81}$ or approximately 0.025.

$y = 3^x$

Homework

base

b) $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$

$a = \frac{1}{2}$ $b = \frac{1}{5}$ $h = 0$ $k = -5$

$(x, y) \longrightarrow (5x, -\frac{1}{2}y - 5)$

i) state the parameters and describe the corresponding transformations

ii) create a table to show what happens to the given points under each transformation

iii) sketch the graph of the base function and the transformed function

iv) describe the effects on the domain, range, equation of the horizontal asymptote, and intercepts

Solution

b) i) Compare the function $y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$ to $y = a(c)^{b(x-h)} + k$ to determine the values of the parameters.

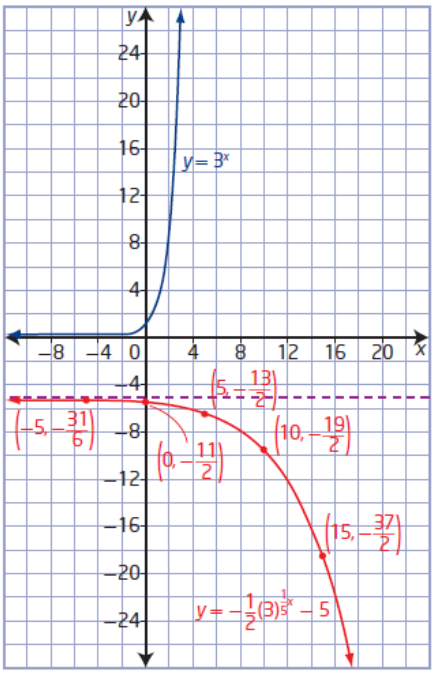
- $b = \frac{1}{5}$ corresponds to a horizontal stretch of factor 5. Multiply the x-coordinates of the points in column 1 by 5.
- $a = \frac{1}{2}$ corresponds to a vertical stretch of factor $\frac{1}{2}$ and a reflection in the x-axis. Multiply the y-coordinates of the points in column 2 by $-\frac{1}{2}$.
- $h = 0$ corresponds to no horizontal translation.
- $k = -5$ corresponds to a translation of 5 units down. Subtract 5 from the y-coordinates of the points in column 3.

ii) Add columns to the table representing the transformations.

$y = 3^x$	$y = 3^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x}$	$y = -\frac{1}{2}(3)^{\frac{1}{5}x} - 5$
$\left(-1, \frac{1}{3}\right)$	$\left(-5, \frac{1}{3}\right)$	$\left(-5, -\frac{1}{6}\right)$	$\left(-5, -\frac{31}{6}\right)$
(0, 1)	(0, 1)	$\left(0, -\frac{1}{2}\right)$	$\left(0, -\frac{11}{2}\right)$
(1, 3)	(5, 3)	$\left(5, -\frac{3}{2}\right)$	$\left(5, -\frac{13}{2}\right)$
(2, 9)	(10, 9)	$\left(10, -\frac{9}{2}\right)$	$\left(10, -\frac{19}{2}\right)$
(3, 27)	(15, 27)	$\left(15, -\frac{27}{2}\right)$	$\left(15, -\frac{37}{2}\right)$

iii) To sketch the graph, plot the points from column 4 and draw a smooth curve through them.

Why do the exponential curves have different horizontal asymptotes?



iv) The domain remains the same: $\{x \mid x \in R\}$.

The range changes to $\{y \mid y < -5, y \in R\}$ because the graph of the transformed function only exists below the line $y = -5$.

The equation of the asymptote changes to $y = -5$.

There is still no x-intercept, but the y-intercept changes to $-\frac{11}{2}$ or -5.5.

Homework

#1-7 and #10 on page 354

7.2 Transformations of Exponential Functions, pages 354 to 357

1. a) C **b)** D **c)** A **d)** B

2. a) D **b)** A **c)** B **d)** C

3. a) $a = 2$: vertical stretch by a factor of 2; $b = 1$: no horizontal stretch; $h = 0$: no horizontal translation; $k = -4$: vertical translation of 4 units down

b) $a = 1$: no vertical stretch; $b = 1$: no horizontal stretch; $h = 2$: horizontal translation of 2 units right; $k = 3$: vertical translation of 3 units up

c) $a = -4$: vertical stretch by a factor of 4 and a reflection in the x-axis; $b = 1$: no horizontal stretch; $h = -5$: horizontal translation of 5 units left; $k = 0$: no vertical translation

d) $a = 1$: no vertical stretch; $b = 3$: horizontal stretch by a factor of $\frac{1}{3}$; $h = 1$: horizontal translation of 1 unit right; $k = 0$: no vertical translation

e) $a = -\frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$ and a reflection in the x-axis; $b = 2$: horizontal stretch by a factor of $\frac{1}{2}$; $h = 4$: horizontal translation of 4 units right; $k = 3$: vertical translation of 3 units up

f) $a = -1$: reflection in the x-axis; $b = 2$: horizontal stretch by a factor of $\frac{1}{2}$; $h = 1$: horizontal translation of 1 unit right; $k = 0$: no vertical translation

g) $a = 1.5$: vertical stretch by a factor of 1.5; $b = \frac{1}{2}$: horizontal stretch by a factor of 2; $h = 4$: horizontal translation of 4 units right; $k = -\frac{5}{2}$: vertical translation of $\frac{5}{2}$ units down

4. a) C: reflection in the x-axis, $a < 0$ and $0 < c < 1$, and vertical translation of 2 units up, $k = 2$

b) A: horizontal translation of 1 unit right, $h = 1$, and vertical translation of 2 units down, $k = -2$

c) D: reflection in the x-axis, $a < 0$ and $c > 1$, and vertical translation of 2 units up, $k = 2$

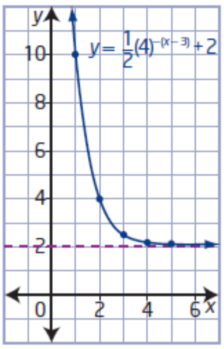
d) B: horizontal translation of 2 units right, $h = 2$, and vertical translation of 1 unit up, $k = 1$

5. a) $a = \frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$; $b = -1$: reflection in the y-axis; $h = 3$: horizontal translation of 3 units right 3; $k = 2$: vertical translation of 2 units up

b)

$y = 4^x$	$y = 4^{-x}$	$y = \frac{1}{2}(4)^{-x}$	$y = \frac{1}{2}(4)^{-(x-3)} + 2$
$\left(-2, \frac{1}{16}\right)$	$\left(2, \frac{1}{16}\right)$	$\left(2, \frac{1}{32}\right)$	$\left(5, \frac{65}{32}\right)$
$\left(-1, \frac{1}{4}\right)$	$\left(1, \frac{1}{4}\right)$	$\left(1, \frac{1}{8}\right)$	$\left(4, \frac{17}{8}\right)$
(0, 1)	(0, 1)	$\left(0, \frac{1}{2}\right)$	$\left(3, \frac{5}{2}\right)$
(1, 4)	(−1, 4)	(−1,2)	(2, 4)
(2, 16)	(−2, 16)	(−2, 8)	(1, 10)

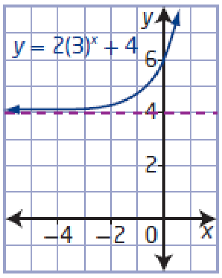
c)



d) domain $\{x \mid x \in R\}$, range $\{y \mid y > 2, y \in R\}$, horizontal asymptote $y = 2$, y-intercept 34

6. a) i), ii) $a = 2$: vertical stretch by a factor of 2; $b = 1$: no horizontal stretch; $h = 0$: no horizontal translation; $k = 4$: vertical translation of 4 units up

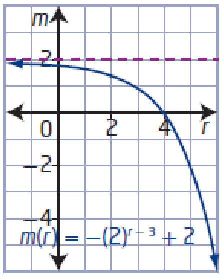
iii)



iv) domain $\{x \mid x \in R\}$, range $\{y \mid y > 4, y \in R\}$, horizontal asymptote $y = 4$, y-intercept 6

b) i), ii) $a = -1$: reflection in the x-axis; $b = 1$: no horizontal stretch; $h = 3$: horizontal translation of 3 units right; $k = 2$: vertical translation of 2 units up

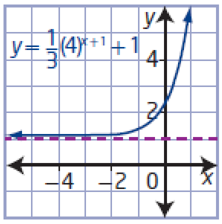
iii)



iv) domain $\{r \mid r \in R\}$, range $\{m \mid m < 2, m \in R\}$, horizontal asymptote $m = 2$, m-intercept $\frac{15}{8}$, r-intercept 4

c) i), ii) $a = \frac{1}{3}$: vertical stretch by a factor of $\frac{1}{3}$; $b = 1$: no horizontal stretch; $h = -1$: horizontal translation of 1 unit left; $k = 1$: vertical translation of 1 unit up

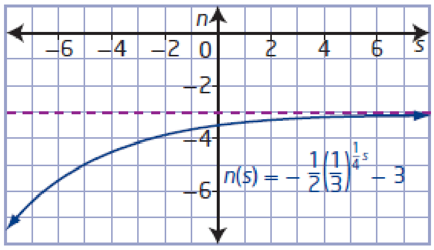
iii)



iv) domain $\{x \mid x \in R\}$, range $\{y \mid y > 1, y \in R\}$, horizontal asymptote $y = 1$, y-intercept $\frac{7}{3}$

d) i), ii) $a = -\frac{1}{2}$: vertical stretch by a factor of $\frac{1}{2}$ and a reflection in the x-axis; $b = \frac{1}{4}$: horizontal stretch by a factor of 4; $h = 0$: no horizontal translation; $k = -3$: vertical translation of 3 units down

iii)



iv) domain $\{s \mid s \in R\}$, range $\{n \mid n < -3, n \in R\}$, horizontal asymptote $n = -3$, n-intercept $-\frac{7}{2}$

7. a) horizontal translation of 2 units right and vertical translation of 1 unit up; $y = \left(\frac{1}{2}\right)^{x-2} + 1$

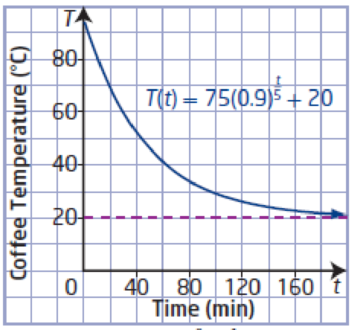
b) reflection in the x-axis, vertical stretch by a factor of 0.5, and horizontal translation of 3 units right; $y = -0.5(5)^{x-3}$

c) reflection in the x-axis, horizontal stretch by a factor of $\frac{1}{3}$, and vertical translation of 1 unit up; $y = -\left(\frac{1}{4}\right)^{3x} + 1$

d) vertical stretch by a factor of 2, reflection in the y-axis, horizontal stretch by a factor of 3, horizontal translation of 1 unit right, and vertical translation of 5 units down; $y = 2(4)^{-\frac{1}{3}(x-1)} - 5$

10. a) $a = 75$: vertical stretch by a factor of 75; $b = \frac{1}{5}$: horizontal stretch by a factor of 5; $h = 0$: no horizontal translation; $k = 20$: vertical translation of 20 units up

b)



c) 29.1 °C **d)** final temperature of the coffee