Understanding Logarithms

$(x,y) \rightarrow (y,x)$
reflection in the line $y=x$

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1.

Logarithmic Form	Exponential Form
$\log_c x = y$	$c^y = x$

exponent

base

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

ans. exp.

a) $32 = 2^5$

base

$\log_2 32 = 5$

b) $2^{-5} = \frac{1}{32}$

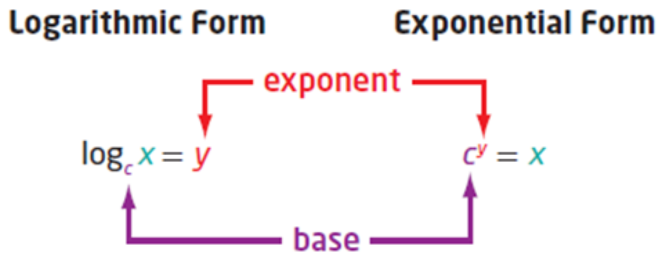
$\log_2\left(\frac{1}{32}\right) = -5$

c) $x = 10^y$

$\log_{10} x = y$

or $\log x = y$

common logarithm



Write each of the following in exponential form

ans. exp

a) $\log_4 16 = 2$

base

$4^2 = 16$

b) $\log_2\left(\frac{1}{32}\right) = -5$

$2^{-5} = \frac{1}{32}$

c) $\log 65 = 1.8129$

$10^{1.8129} = 65$

Example 1

Evaluating a Logarithm

Evaluate.

a) $\log_7 49$ **b)** $\log_6 1$ **c)** $\log 0.001$ **d)** $\log_2 \sqrt{8}$

a) Let $x = \log_7 49$

$x = \log_7 49$

$7^x = 49$

$7^x = 7^2$

$x = 2$

$\log_7 49 = 2$

b) $x = \log_6 1$

$6^x = 1$

$6^x = 6^0$

$x = 0$

$\log_6 1 = 0$

c) $x = \log 0.001$

$10^x = 0.001$

$10^x = 10^{-3}$

$x = -3$

$\log 0.001 = -3$

d) $x = \log_2 \sqrt{8}$

$2^x = \sqrt{8}$

$2^x = 8^{1/2}$

$2^x = (2^3)^{1/2}$

$2^x = 2^{3/2}$

$x = 3/2$

$\log_2 \sqrt{8} = \frac{3}{2}$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

a) $\log_5 x = -3$

$5^{-3} = x$

$\left(\frac{1}{5}\right)^3 = x$

$\frac{1}{125} = x$

b) $\log_x 36 = 2$

$x^2 = 36$

$x = \pm 6$

$x = 6$

Choose $x = 6$ because your base has to be greater than 0

c) $\log_{64} x = \frac{2}{3}$

$64^{2/3} = x$

$\left(\sqrt[3]{64}\right)^2 = x$

$(4)^2 = x$

$16 = x$

Questions from Homework

4 c) $\log_4 \sqrt[3]{4}$

$\log_4 4^{1/3} = x$

$\cancel{4}^x = \cancel{4}^{1/3}$

$\boxed{x = \frac{1}{3}}$

d) $\log_{\frac{1}{3}} 27$

$\log_{\frac{1}{3}} 27 = x$

$\left(\frac{1}{3}\right)^x = 27$

$(3^{-1})^x = 3^3$

$\cancel{3}^{-x} = \cancel{3}^3$

$-x = 3$

$\boxed{x = -3}$

General Properties of Logarithms:

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:

- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

a) Find the inverse.

$f(x) = 3^x$

① $y = 3^x$

② $x = 3^y$

③ $y = \log_3 x$

④ $f^{-1}(x) = \log_3 x$

Solution

a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $y = \log_3 x$. Since the inverse is a function, it can be written in function notation as $f^{-1}(x) = \log_3 x$.

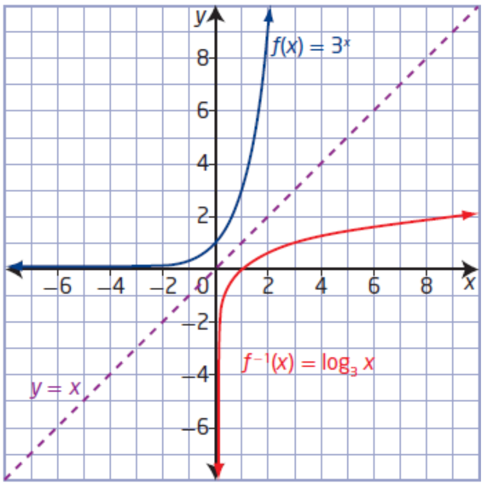
How do you know that $y = \log_3 x$ is a function?

b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$(x,y) \rightarrow (y,x)$

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in R\}$ and the range is $\{y \mid y \in R\}$
- the x-intercept is 1
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

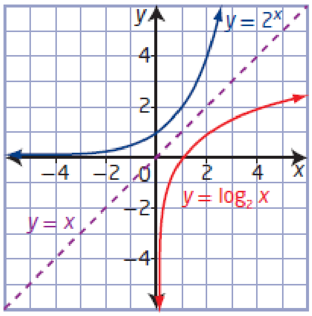
How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$
  - the range is $\{y \mid y \in \mathrm{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

$$\log_{10} x = \log x$$

Homework

#1-5, 8, 10, 12, 13, 17 on page 380

17 $N(t) = 1.1^t \rightarrow f(x) = 1.1^x$

a) Find the inverse

1 $y = 1.1^x$

2 $x = 1.1^y$

3 $\log_{1.1} x = y$ or $y = \log_{1.1} x$

4 $f^{-1}(x) = \log_{1.1} x$

$$\boxed{N^{-1}(t) = \log_{1.1} t}$$

b) $N(t) = 1.1^t$ Given: $N(t) = 1000000$

$1000000 = 1.1^t$

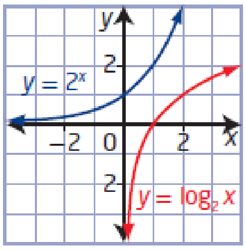
$\frac{\log 1000000}{\log 1.1} = 145$

$(\cancel{1.1})^{145} = \cancel{1.1}^t$

145 days $= t$

8.1 Understanding Logarithms, pages 380 to 382

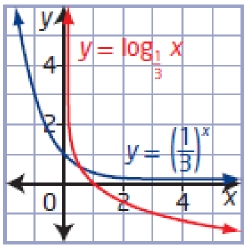
1. a) i)



ii) $y = \log_2 x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

b) i)



ii) $y = \log_{\frac{1}{3}} x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

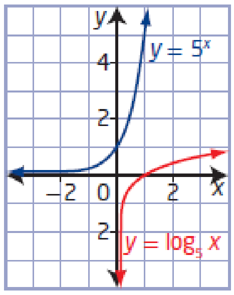
2. a) $\log_{12} 144 = 2$ **b)** $\log_8 2 = \frac{1}{3}$
c) $\log_{10} 0.000\,01 = -5$ **d)** $\log_7 (y + 3) = 2x$

3. a) $5^2 = 25$ **b)** $8^{\frac{2}{3}} = 4$
c) $10^6 = 1\,000\,000$ **d)** $11^y = x + 3$

4. a) 3 **b)** 0 **c)** $\frac{1}{3}$ **d)** -3

5. $a = 4$; $b = 5$

8. a) $y = \log_5 x$
b)



domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

10. They are reflections of each other in the line $y = x$.

11. a) They have the exact same shape.
b) One of them is increasing and the other is decreasing.

12. a) 216 **b)** 81 **c)** 64 **d)** 8

13. a) 7 **b)** 6

14. a) 0 **b)** 1

15. -1

16. 16

17. a) $t = \log_{1.1} N$ **b)** 145 days

18. The larger asteroid had a relative risk that was 1479 times as dangerous.

19. 1000 times as great

20. 5

21. $m = 14$, $n = 13$

22. $4n$

23. $y = 3^{2^x}$

Product Law of Logarithms

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$

$MN = c^{x+y}$ Apply the product law of powers.

$\log_c MN = x + y$ Write in logarithmic form.

$\log_c MN = \log_c M + \log_c N$ Substitute for x and y.

Ex: $\log_2 x^2 y = \log_2 x^2 + \log_2 y$

$\log_3 27 + \log_3 3 = \log_3 (27 \times 3) = \log_3 81 = 4$

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$ Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$ Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$ Substitute for x and y.

$$\log_5 50 - \log_5 2 = \log_5\left(\frac{50}{2}\right) = \log_5 25 = 2$$

$$\log_5 50 + \log_5 2 = \log_5(50\times 2) = \underline{\log_5 100} \approx 2.86$$

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.

Let P be a real number.

$$M = c^x$$
$$M^P = (c^x)^P$$
$$M^P = c^{xP} \quad \text{Simplify the exponents.}$$
$$\log_c M^P = xP \quad \text{Write in logarithmic form.}$$
$$\log_c M^P = (\log_c M)P \quad \text{Substitute for } x.$$
$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Ex: $\log_5 \sqrt[3]{125} \rightarrow \log_5 5$

1

$\log_5 (125)^{\frac{1}{3}}$

$\frac{1}{3} \log_5 125$

$\frac{1}{3}(3)$

1

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z} = \log_5 x + \log_5 y - \log_5 z$

b) $\log_7 \sqrt[3]{x} = \log_7 x^{\frac{1}{3}} = \frac{1}{3}\log_7 x$

c) $\log_6 \frac{1}{x^2} = \log_6 x^{-2} = -2\log_6 x$

d) $\log \frac{x^3}{y\sqrt{z}} = \log x^3 - [\log y\sqrt{z}]$

$\log x^3 - [\log y + \log z^{\frac{1}{2}}]$

$\log x^3 - \log y - \log z^{\frac{1}{2}}$

$3\log x - \log y - \frac{1}{2}\log z$

$= \log x^3 - \log y - \log\sqrt{z}$

$= 3\log x - \log y - \frac{1}{2}\log z$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$ $= \log_6\left(\frac{8 \cdot 9}{2}\right) = \log_6 36 = 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

b) $\log_7 7\sqrt{7}$

$= \log_7 7 + \log_7 \sqrt{7}$

$= 1 + \frac{1}{2}\boxed{\log_7 7}$

$= 1 + \frac{1}{2}(1)$

$= \frac{3}{2}$

c) $2\log_2 12 - \left(\log_2 6 + \frac{1}{3}\log_2 27\right)$

$= \log_2 12^2 - \left(\log_2 6 + \log_2 27^{1/3}\right)$

$= \log_2 144 - \left(\log_2 6 + \log_2 3\right)$

$= \log_2 144 - \log_2 18$

$= \log_2 8$

$= 3$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

a) $\log_7 x^2 + \log_7 x - \frac{5}{2}\log_7 x$

$= \log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$= \log_7 \left(\frac{x^2 \cdot x}{x^{5/2}}\right)$

$= \log_7 \left(\frac{x^3}{x^{5/2}}\right)$

$\frac{x^3}{x^{5/2}} = x^{3-\frac{5}{2}} = x^{\frac{6}{2}-\frac{5}{2}} = x^{1/2}$

$= \log_7 x^{1/2}$

$= \frac{1}{2}\log_7 x$

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12(\log_b x^2 - 2\log_b x) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$