

Introduction to Trigonometric Equations

trigonometric equation

- an equation involving trigonometric ratios

Focus on...

- algebraically solving first-degree and second-degree trigonometric equations in radians and in degrees
- verifying that a specific value is a solution to a trigonometric equation
- identifying exact and approximate solutions of a trigonometric equation in a restricted domain
- determining the general solution of a trigonometric equation

Did You Know?

In equations, mathematicians often use the notation $\cos^2 \theta$. This means the same as $(\cos \theta)^2$.

Let's start with basic LINEAR trigonometric equations...

...Pre-Calculus 110

Solve: $\sin \theta = 0.9659$, $-360^\circ < \theta < 720^\circ$

- Reference angle?
- Which quadrants?
- Any co-terminal angles acceptable?

- If the domain is in degrees, give solutions in degrees.
- If the domain is in radians, give solutions in radians.

$\sin \theta = 0.9659$, $-360^\circ < \theta < 720^\circ$ Where is $\sin \theta$ positive

$$\bar{\theta} = \sin^{-1}(0.9659)$$

$$\bar{\theta} = 75^\circ$$

↑
reference angle

Q1	Q2
$\theta = \bar{\theta}$	$\theta = 180^\circ - \bar{\theta}$
$\theta = 75^\circ$	$\theta = 180^\circ - 75^\circ$
	$\theta = 105^\circ$

To find angles between -360° and 720°

$$\begin{array}{l|l} 75^\circ - 360^\circ = -285^\circ & 75^\circ + 360^\circ = 435^\circ \\ 105^\circ - 360^\circ = -255^\circ & 105^\circ + 360^\circ = 465^\circ \end{array}$$

Solutions:

$$\theta = -285^\circ, -255^\circ, 75^\circ, 105^\circ, 435^\circ, 465^\circ$$

Solve: $\sec \theta = -1.3054$, $-2\pi \leq \theta \leq 2\pi$ (radians)

(reciprocal)
 $\cos \theta = \frac{1}{-1.3054}$

$$\cos \theta = -0.7660$$

$$\bar{\theta} = \cos^{-1}(0.7660)$$

$$\bar{\theta} = 0.698$$

Where is $\cos \theta$ negative

Q2	Q3
$\theta = \pi - \bar{\theta}$	$\theta = \pi + \bar{\theta}$
$\theta = 3.14 - 0.698$	$\theta = 3.14 + 0.698$
$\theta = 2.443$	$\theta = 3.838$

Find angles between -2π and 2π (-6.28) (6.28)

$$2.443 - 6.28 = -3.838$$

$$3.838 - 6.28 = -2.443$$

Solutions:

$$\theta = -3.838, -2.443, 2.443, 3.838$$

Exact Values \rightarrow No Calculators

Ex. $\sqrt{2} \cos \theta + 1 = 0, -360^\circ \leq \theta \leq 720^\circ$

$$\sqrt{2} \cos \theta = -1$$

$$\cos \theta = \frac{-1}{\sqrt{2}}$$

(Use Special Triangles) $\bar{\theta} = 45^\circ$

Where is $\cos \theta$ negative

Q2	Q3
$\theta = 180^\circ - \bar{\theta}$	$\theta = 180^\circ + \bar{\theta}$
$\theta = 180^\circ - 45^\circ$	$\theta = 180^\circ + 45^\circ$
$\theta = 135^\circ$	$\theta = 225^\circ$

Find angles between -360° and 720°

$$135^\circ - 360^\circ = -225^\circ$$

$$225^\circ - 360^\circ = -135^\circ$$

$$135^\circ + 360^\circ = 495^\circ$$

$$225^\circ + 360^\circ = 585^\circ$$

Solutions.

$$\theta = -225^\circ, -135^\circ, 135^\circ, 225^\circ, 495^\circ, 585^\circ$$

Exact Value \rightarrow No Calculator

Ex. $\sin x + 1 = 0, -2\pi \leq x \leq 4\pi$

$$\sin x = -1$$

(Unit Circle) $x = \frac{3\pi}{2}$

$$\frac{3\pi}{2} - 2\pi$$

$$\frac{3\pi}{2} - \frac{4\pi}{2}$$

$$-\frac{\pi}{2}$$

$$\frac{3\pi}{2} + 2\pi$$

$$\frac{3\pi}{2} + \frac{4\pi}{2}$$

$$\frac{7\pi}{2}$$

Solutions:

$$x = -\frac{\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$$

Your Turn

Solve each trigonometric equation in the specified domain.

a) $3 \cos \theta - 1 = \cos \theta + 1, -2\pi \leq \theta \leq 2\pi$

b) $4 \sec x + 8 = 0, 0^\circ \leq x < 360^\circ$

Check Up:

$$\cot \theta = 0.7834 \quad 0 < \theta < 2\pi$$

radians

$$\tan \theta = 1.2765$$

$$\bar{\theta} = \tan^{-1}(1.2765)$$

$$\bar{\theta} = 0.9063$$

Where is $\tan \theta$ positive

Q1	Q3
$\theta = \bar{\theta}$	$\theta = \pi + \bar{\theta}$
$\theta = 0.9063$	$\theta = 3.14 + 0.9063$
	$\theta = 4.0463$

Solutions:

$$\theta = 0.9063, 4.0463$$

$$2\cos \theta - 5 = -6 \quad -360^\circ < \theta < 360^\circ$$

Degrees

$$2\cos \theta = -1$$

$$\cos \theta = -\frac{1}{2}$$

$$\bar{\theta} = 60^\circ$$

Where is $\cos \theta$ negative?

Q2	Q3
$\theta = 180^\circ - \bar{\theta}$	$\theta = 180^\circ + \bar{\theta}$
$\theta = 180^\circ - 60^\circ$	$\theta = 180^\circ + 60^\circ$
$\theta = 120^\circ$	$\theta = 240^\circ$

Find angles between -360° and 360°

$$120^\circ - 360^\circ = -240^\circ$$

$$240^\circ - 360^\circ = -120^\circ$$

Solutions:

$$\theta = -240^\circ, -120^\circ, 120^\circ, 240^\circ$$

$$\textcircled{3} \text{ c) } 5 - \tan^2 \theta = 4, \quad -180^\circ \leq \theta \leq 360^\circ$$

$$-\tan^2 \theta = -1$$

$$\tan^2 \theta = 1$$

$$\tan \theta = \pm 1$$

$$\bar{\theta} = 45^\circ$$

where is $\tan \theta$ (+/-)

Q1	Q2	Q3	Q4
$\theta = 45^\circ$	$\theta = 135^\circ$	$\theta = 225^\circ$	$\theta = 315^\circ$
$\theta = 315^\circ$	$\theta = 225^\circ$	$\theta = -135^\circ$	$\theta = -45^\circ$

Solutions:

$$\theta = -135^\circ, -45^\circ, 45^\circ, 135^\circ, 225^\circ, 315^\circ$$

Questions from Homework

$$\textcircled{11} \cos \theta = -0.15$$

$$\bar{\theta} = \cos^{-1}(0.15) \text{ when finding } \bar{\theta} \text{ always use positive value}$$

$$\bar{\theta} = 81.4^\circ$$

$\cos^{-1}(0.15)$ 81.37307344

Where is $\cos \theta$ **negative**

Q2	Q3
$\theta = 180^\circ - 81.4^\circ$	$\theta = 180^\circ + 81.4^\circ$
$\theta = 98.6^\circ$	$\theta = 261.4^\circ$

$$98.6^\circ + 360^\circ n, n \in \mathbb{I}$$

$$261.4^\circ + 360^\circ n, n \in \mathbb{I}$$

Let's move onto QUADRATIC trigonometric equations...

...Pre-Calculus 110

\uparrow (2nd Degree) (Factoring!!)

- What strategies can we use to solve quadratic equations?
- Quadratic trigonometric equations will ultimately become TWO linear trigonometric equations.

Solve: $2x^2 + x = 1$

$$2x^2 + x - 1 = 0$$

$$(2x^2 + 2x)(x - 1) = 0$$

$$2x(x+1) - 1(x+1) = 0$$

$$(2x-1)(x+1) = 0$$

$2x-1=0$	$x+1=0$
$2x=1$	$x=-1$
$x=\frac{1}{2}$	

$$2x-1 = -2$$

$$2+1 = 1$$

Solve: $2\sin^2 x + \sin x = 1, 0 \leq x \leq 4\pi$

$$2\sin^2 x + \sin x - 1 = 0$$

$$(2\sin^2 x + 2\sin x)(\sin x - 1) = 0$$

$$2\sin x(\sin x + 1) - 1(\sin x + 1) = 0$$

$$(2\sin x - 1)(\sin x + 1) = 0$$

$2\sin x - 1 = 0$	$\sin x + 1 = 0$
$\sin x = \frac{1}{2}$	$\sin x = -1$
(Triangles) $x = \frac{\pi}{6}$	(Unit Circle) $x = \frac{3\pi}{2}$

Q1 $x = \frac{\pi}{6}$	Q2 $x = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$	$x = \frac{7\pi}{2}$
$x = \frac{13\pi}{6}$	$x = \frac{17\pi}{6}$	

Solutions

$$x = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{3\pi}{2}, \frac{7\pi}{2}$$

Ex. $\cos^2 \theta - \frac{1}{2} \cos \theta = 0, -2\pi \leq \theta \leq 4\pi$

$$\cos \theta \left(\cos \theta - \frac{1}{2} \right) = 0 \quad (\text{Common Factor})$$

$$\cos \theta = 0$$

(Unit Circle)

$$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$$

$$\theta = \frac{5\pi}{2}, \frac{7\pi}{2}$$

$$\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}$$

$$\cos \theta - \frac{1}{2} = 0$$

$$\cos \theta = \frac{1}{2}$$

(Triangles)

$$\theta = \frac{\pi}{3}$$

Q1

$$\theta = \frac{\pi}{3}$$

$$\theta = \frac{7\pi}{3}$$

$$\theta = -\frac{5\pi}{3}$$

Q4

$$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$$

$$\theta = \frac{11\pi}{3}$$

$$\theta = -\frac{\pi}{3}$$

Ex. $6 \sin^2 x - \sin x = 2, -2\pi \leq \theta \leq 4\pi$ (Decomposition)

General Solution of a Trigonometric Equation

Solve: $3\cos^2 \theta - \cos \theta = 2; \theta \in \mathbb{R}$ (Decomposition)

Determine the general solution for $\sin^2 x - 1 = 0$ over the real numbers if x is measured in radians.

$$\begin{aligned} \sin^2 x - 1 &= 0 \\ (\sin x + 1)(\sin x - 1) &= 0 \\ \sin x + 1 = 0 & \quad | \quad \sin x - 1 = 0 \\ \sin x = -1 & \quad | \quad \sin x = 1 \\ \text{(unit circle)} & \quad | \quad \text{(unit circle)} \\ \theta = \frac{3\pi}{2} & \quad | \quad \theta = \frac{\pi}{2} \end{aligned}$$

(Difference of Squares)

Did You Know?

$2n$, where $n \in \mathbb{I}$, represents all even integers.

$2n + 1$, where $n \in \mathbb{I}$, is an expression for all odd integers.

$$x = \frac{\pi}{2} + 2\pi n, \text{ where } n \in \mathbb{I}$$

$$x = \frac{3\pi}{2} + 2\pi n, \text{ where } n \in \mathbb{I}$$

or

$$x = \frac{\pi}{2} + \pi n, \text{ where } n \in \mathbb{I}$$

or

$$(2n + 1)\left(\frac{\pi}{2}\right), n \in \mathbb{I}$$

Determine the general solution for $\cos^2 x - 1 = 0$, where the domain is real numbers measured in degrees.

Practice Problems:

Pages 212 - 214

#11-13, 16, 22

Check-Up problem...

Solve:

$$\sin x \sec x + 2 \sin x = 0, \quad x \in R \quad (x \text{ is measured in radians})$$

Unit Review...

What topics have we covered??

Review...

- C4 a)** Determine all solutions for the equation $2 \sin^2 \theta = 1 - \sin \theta$ in the domain $0^\circ \leq \theta < 360^\circ$.
- b)** Are your solutions exact or approximate? Why?
- c)** Show how you can check one of your solutions to verify its correctness.

A grandfather clock shows a time of 7 o'clock. What is the exact radian measure of the angle between the hour hand and the minute hand?

Determine the angular velocity of the minute hand on a clock.

Solve: $6 \sin^2 \theta - 3 \sin \theta = 0$, $0 \leq \theta \leq 360^\circ$

[A] $0^\circ, 30^\circ, 180^\circ, 330^\circ, 360^\circ$

[B] $0^\circ, 30^\circ, 180^\circ, 150^\circ, 360^\circ$

[C] $30^\circ, 90^\circ, 120^\circ, 270^\circ$

[D] $0^\circ, 180^\circ, 210^\circ, 330^\circ, 360^\circ$

If $\csc \theta < 0$ and $\tan \theta > 0$, then which of the following could be a possible measure of angle θ ?

[A] $\frac{11\pi}{6}$

[B] $\frac{4\pi}{3}$

[C] $\frac{3\pi}{4}$

[D] $\frac{\pi}{2}$

What is the principal angle of $-\frac{25\pi}{4}$?

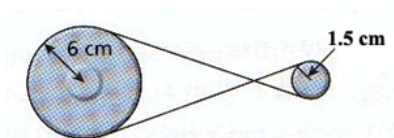
[A] $\frac{3\pi}{4}$

[B] $\frac{\pi}{4}$

[C] $-\frac{\pi}{4}$

[D] $\frac{7\pi}{4}$

If the belt in the pulley system below travels 30 cm, what is the angle of rotation of the smaller pulley?



[A] $\frac{\pi}{9}$ radians

[B] 20°

[C] 20 radians

[D] 5°

Nibbles the hamster is running at 0.02 m/s on an exercise wheel of radius 8 cm. What is the angular velocity of this wheel?

[A] 0.15 rad/minute

[B] 240 rad/minute

[C] 0.25 rad/minute

[D] 15 radians/minute

Solve: $2(1 - \sin \theta)^2 + \sin \theta = 2(3 - 4 \sin^2 \theta)$, $-360^\circ \leq \theta \leq 720^\circ$

Little Johnny has a rock tied to the end of a piece of rope 1.5 m long and he is swinging it around his head in a circular pattern. Mrs. Centripetal, his physics teacher, is watching Johnny out the window of her physics lab and notes that the rock is making 12 revolutions every 48 seconds.

- (a) Determine the angular velocity with which little Johnny is twirling the rope above his head. [2]

- (b) The rock comes flying from the rope 3 minutes after Mrs. Centripetal started to time little Johnny. How far did the rock travel during the 3 minutes? [2]

Attachments

Worksheet - Sketching Angles in Radians.doc