

Understanding Logarithms

Chapter 8 (page 370)

Focus on...

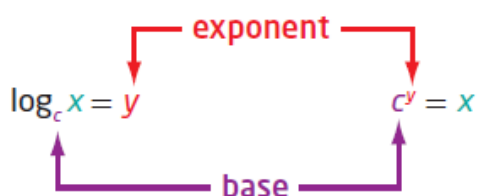
$(x,y) \rightarrow (y,x)$
reflection in the line $y=x$

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1. (Page 373)

Logarithmic Form

Exponential Form



Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a **common logarithm**, you do not need to write the base. For example, **log 3** means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

a) $32 = 2^5$ ← exp.

↑ answer.
↑ base

$\log_2(32) = 5$

$5 = \log_2 32$

b) $2^{-5} = \frac{1}{32}$

$\log_2\left(\frac{1}{32}\right) = -5$

c) $x = 10^y$

$\log_{10} x = y$

$\log x = y$

Logarithmic Form

Exponential Form



Write each of the following in exponential form

a) $\log_4 16 = 2$ ← exp.

↑ base
↑ answer

$4^2 = 16$

$16 = 4^2$

b) $\log_2\left(\frac{1}{32}\right) = -5$

$2^{-5} = \frac{1}{32}$

c) $\log 65 = 1.8129$

$10^{1.8129} = 65$

Example 1

Evaluating a Logarithm

Evaluate. (Solving for an exponent)

a) $\log_7 49$

b) $\log_6 1$

c) $\log 0.001$

d) $\log_2 \sqrt{8}$

a) Let $x = \log_7 49$

$$x = \log_7 49$$

$$7^x = 49 \leftarrow \begin{array}{l} \text{express} \\ \text{in exponential} \\ \text{form} \end{array}$$

$$7^x = 7^2 \leftarrow \begin{array}{l} \text{get common} \\ \text{base} \end{array}$$

$$x = 2$$

$$\therefore \log_7 49 = 2$$

b) Let $x = \log_6 1$

$$x = \log_6 1$$

$$6^x = 1$$

$$6^x = 6^0$$

$$x = 0$$

$$\therefore \log_6 1 = 0$$

c) $x = \log 0.001$

$$10^x = 0.001$$

$$10^x = 10^{-3}$$

$$x = -3$$

$$\therefore \log 0.001 = -3$$

d) $x = \log_2 \sqrt{8}$

$$2^x = \sqrt{8}$$

$$2^x = (8)^{1/2}$$

$$2^x = (2^3)^{1/2}$$

$$2^x = 2^{3/2}$$

$$x = \frac{3}{2}$$

$$\therefore \log_2 \sqrt{8} = \frac{3}{2}$$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x .

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

a) $\log_5 x = -3$

$$5^{-3} = x$$

$$\left(\frac{1}{5}\right)^3 = x$$

$$\boxed{\frac{1}{125} = x}$$

b) $\log_x 36 = 2$

$$x^2 = 36$$

$$x = \pm 6$$

$$\boxed{\text{Choose } x = 6}$$

c) $\log_{64} x = \frac{2}{3}$

$$(64)^{\frac{2}{3}} = x$$

$$\boxed{16 = x}$$

Example 3

Graph the Inverse of an Exponential Function

- a) State the inverse of $f(x) = 3^x$.
- b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:
- the domain and range
 - the x-intercept, if it exists
 - the y-intercept, if it exists
 - the equations of any asymptotes

To Find Inverse:

$$a) f(x) = 3^x$$

$$y = 3^x \quad (\text{Replace } f(x) \text{ with } y)$$

$$x = 3^y \quad (\text{Switch } x \leftrightarrow y)$$

$$\boxed{y = \log_3 x} \quad (\text{Solve for } y) \rightarrow \text{Express in logarithmic form}$$

$$\text{or } f^{-1}(x) = \log_3 x$$

$$f(x) = 3^x \longrightarrow f^{-1}(x) = \log_3 x$$

- | | |
|---|---|
| • D: $\{x \mid x \in \mathbb{R}\}$ | • D: $\{x \mid x > 0, x \in \mathbb{R}\}$ |
| • R: $\{y \mid y > 0, y \in \mathbb{R}\}$ | • R: $\{y \mid y \in \mathbb{R}\}$ |
| • x-int: none | • x-int: $(1, 0)$ |
| • y-int: $(0, 1)$ | • y-int: none |
| • HA: $y = 0$ | • VA: $x = 0$ |

Solution

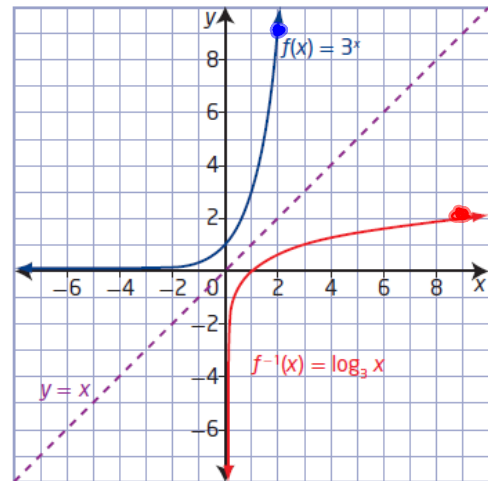
a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $y = \log_3 x$. Since the inverse is a function, it can be written in function notation as $f^{-1}(x) = \log_3 x$.

How do you know that $y = \log_3 x$ is a function? $y = 3^x$ passes the horizontal line test

b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph

of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in \mathbb{R}\}$ and the range is $\{y \mid y \in \mathbb{R}\}$
- the x-intercept is 1 $(1, 0)$
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

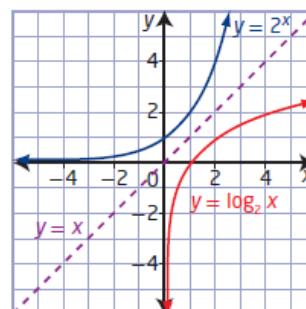
- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form **Logarithmic Form**

$$x = c^y \qquad y = \log_c x$$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.
- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
 - the domain is $\{x \mid x > 0, x \in \mathbb{R}\}$
 - the range is $\{y \mid y \in \mathbb{R}\}$
 - the x-intercept is 1
 - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

$$\log_{10} x = \log x$$



Questions from Homework

8. a) If $f(x) = 5^x$, state the equation of the inverse, $f^{-1}(x)$.
- b) Sketch the graph of $f(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
 - the x-intercept, if it exists
 - the y-intercept, if it exists
 - the equations of any asymptotes

a) Inverse: $f(x) = 5^x$

$y = 5^x$

$x = 5^y$

$y = \log_5 x$

$f^{-1}(x) = \log_5 x$

For the curve
 $f(x) = 5^x$
 D: $\{x | x \in \mathbb{R}\}$
 R: $\{y | y > 0, y \in \mathbb{R}\}$
 x-int: none
 y-int: $(0, 1)$
 HA: $y = 0$

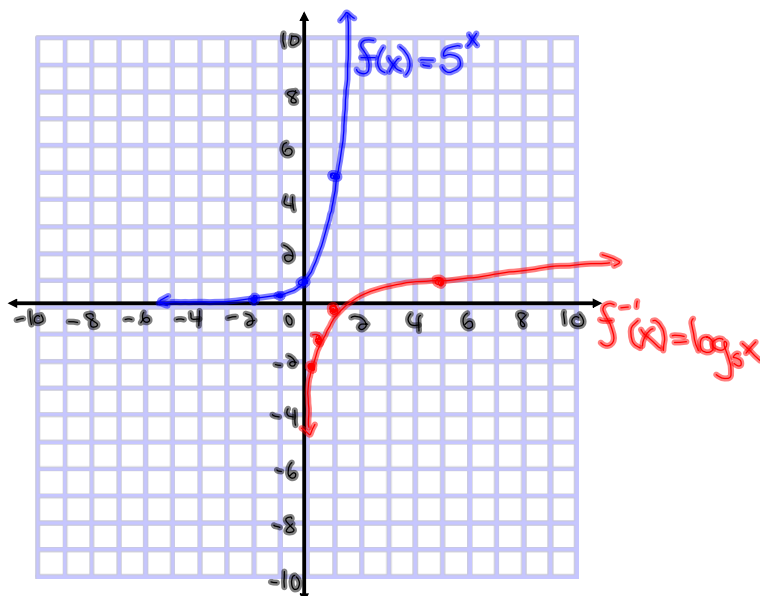
$f(x) = 5^x$

x	y
-2	$\frac{1}{25}$
-1	$\frac{1}{5}$
0	1
1	5
2	25

For the curve
 $f^{-1}(x) = \log_5 x$
 D: $\{x | x > 0, x \in \mathbb{R}\}$
 R: $\{y | y \in \mathbb{R}\}$
 x-int: $(1, 0)$
 y-int: none
 VA: $x = 0$

$f^{-1}(x) = \log_5 x$

x	y
$\frac{1}{25}$	-2
$\frac{1}{5}$	-1
1	0
5	1
25	2



17. The growth of a new social networking site can be modelled by the exponential function $N(t) = 1.1^t$, where N is the number of users after t days.
- Write the equation of the inverse.
 - How long will it take, to the nearest day, for the number of users to exceed 1 000 000?

$$a) N(t) = 1.1^t$$

$$f(x) = 1.1^x$$

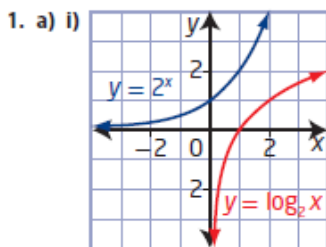
$$y = 1.1^x$$

$$x = 1.1^y$$

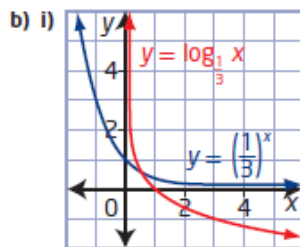
$$y = \log_{1.1} x$$

$$f^{-1}(x) = \log_{1.1} x$$

8.1 Understanding Logarithms, pages 380 to 382

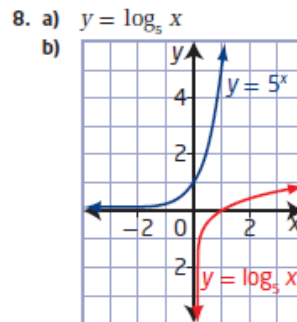


ii) $y = \log_2 x$
 iii) domain $\{x \mid x > 0, x \in \mathbb{R}\}$,
 range $\{y \mid y \in \mathbb{R}\}$,
 x-intercept 1, no y-intercept,
 vertical asymptote $x = 0$



ii) $y = \log_3 x$
 iii) domain $\{x \mid x > 0, x \in \mathbb{R}\}$,
 range $\{y \mid y \in \mathbb{R}\}$,
 x-intercept 1, no y-intercept,
 vertical asymptote $x = 0$

2. a) $\log_{12} 144 = 2$ b) $\log_8 2 = \frac{1}{3}$
 c) $\log_{10} 0.000\ 01 = -5$ d) $\log_7 (y + 3) = 2x$
3. a) $5^2 = 25$ b) $8^{\frac{2}{3}} = 4$
 c) $10^6 = 1\ 000\ 000$ d) $11^y = x + 3$
4. a) 3 b) 0 c) $\frac{1}{3}$ d) -3
5. $a = 4; b = 5$



domain $\{x \mid x > 0, x \in \mathbb{R}\}$,
 range $\{y \mid y \in \mathbb{R}\}$,
 x-intercept 1,
 no y-intercept,
 vertical asymptote $x = 0$

10. They are reflections of each other in the line $y = x$.
11. a) They have the exact same shape.
 b) One of them is increasing and the other is decreasing.
12. a) 216 b) 81 c) 64 d) 8
13. a) 7 b) 6
14. a) 0 b) 1
15. -1
16. 16
17. a) $t = \log_{0.11} N$ b) 145 days
18. The larger asteroid had a relative risk that was 1479 times as dangerous.
19. 1000 times as great
20. 5
21. $m = 14, n = 13$
22. $4n$
23. $y = 3^{2^x}$

Transformations of Logarithmic Functions

Focus on...

- explaining the effects of the parameters a , b , h , and k in $y = a \log_c (b(x - h)) + k$ on the graph of $y = \log_c x$, where $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, where $c > 1$, and stating the characteristics of the graph

Remember:

Parameter	Transformation
a	$(x, y) \rightarrow (x, ay)$
b	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
h	$(x, y) \rightarrow (x + h, y)$
k	$(x, y) \rightarrow (x, y + k)$

Example 1

Translations of a Logarithmic Function

- a) Use transformations to sketch the graph of the function
 $y = \log_3(x + 9) + 2$.
- b) Identify the following characteristics of the graph of the function.
- the equation of the asymptote
 - the domain and range
 - the y-intercept, if it exists
 - the x-intercept, if it exists

$$a) y = \log_3(x+9) + 2$$

$$a=1 \quad b=1 \quad h=-9 \quad k=2$$

- Translated 9 to the left and 2 up

$$\text{Mapping: } (x, y) \rightarrow (x-9, y+2)$$

$$f(x) = 3^x$$

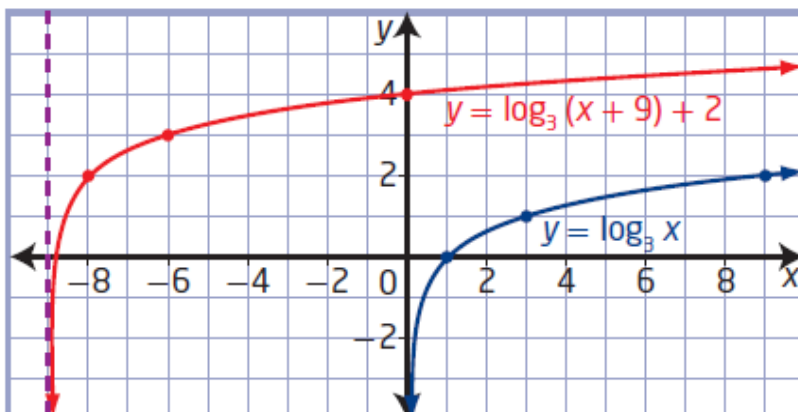
x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

$$y = \log_3 x$$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

$$y = \log_3(x+9) + 2$$

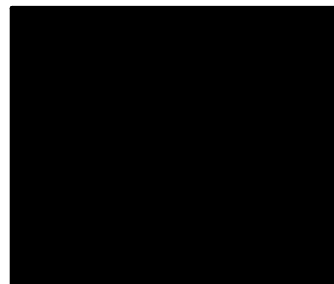
x	y
-8 1/9	0
-6 2/3	1
-8	2
-6	3
0	4



Example 2

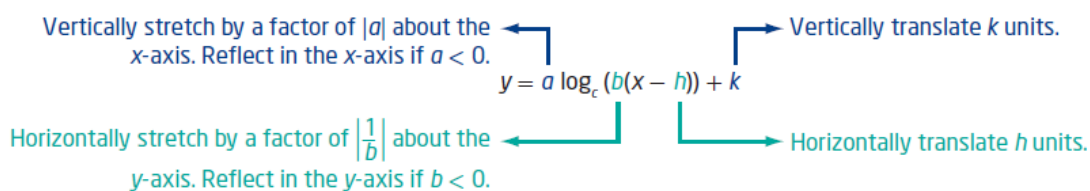
Reflections, Stretches, and Translations of a Logarithmic Function

- a) Use transformations to sketch the graph of the function
 $y = -\log_2(2x + 6)$.
- b) Identify the following characteristics of the graph of the function.
- the equation of the asymptote
 - the domain and range
 - the y -intercept, if it exists
 - the x -intercept, if it exists



Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a , b , h , and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.



- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Homework

