Combining Functions

- Two functions f and g can be combined to form new functions
  - $f+g$,
  - $f-g$,
  - fg , and
  - f/g

  just as we add, subtract, multiply, and divide real numbers.

- This is summarized in the following table:

Algebra of Functions Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$$(f+g)(x) = f(x) + g(x) \qquad \text{domain} = A \cap B$$

$$(f-g)(x) = f(x) - g(x) \qquad \text{domain} = A \cap B$$

$$(fg)(x) = f(x)g(x) \qquad \text{domain} = A \cap B$$

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)} \qquad \text{domain} = \{x \in A \cap B \mid g(x) \neq 0\}$$

Example

- If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{4-x^2}$, find the functions $f+g$, $f-g$, fg, and f/g.

**Also examine the domain of each of these new functions

Compositions of Functions

(combining functions in a different way)

When the input in a function is another function, the result is called a **composite function.** If

$f(x) = 3x + 2$ and $g(x) = 4x - 5$

then $f[g(x)]$ is a composite function. The statement $f[g(x)]$ is read " f of g of x" or "the composition of f with g." $f[g(x)]$ can also be written as

$$(f \circ g)(x) \quad \text{or} \quad f \circ g(x)$$

The symbol between f and g is a small open circle. When replacing one function with another, be very careful to get the order correct because compositions of functions are not necessarily commutative (as you'll see).

f composed with g

$(f \circ g)(x)$

$= f(g(x))$

multiply

$(f \cdot g)(x)$

$= f(x) \cdot g(x)$

Example 1

Evaluate a Composite Function

If $f(x) = 4x$, $g(x) = x + 6$, and $h(x) = x^2$, determine each value.

a) $f(g(3))$

b) $g(h(-2))$

c) $h(h(2))$

Method 1: Determine the Value of the Inner Function and Then Substitute

a) $f(g(3))$

$g(3) = (3) + 6 = 9$

$f(9) = 4(9) = \boxed{36}$

b) $g(h(-2))$

$h(-2) = (-2)^2 = 4$

$g(4) = (4) + 6 = \boxed{10}$

c) $h(h(2))$

$h(2) = (2)^2 = 4$

$h(4) = (4)^2 = \boxed{16}$

Example 2

If $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x - 5$, find each of the following:

1. $f[g(x)]$

2. $g[f(x)]$

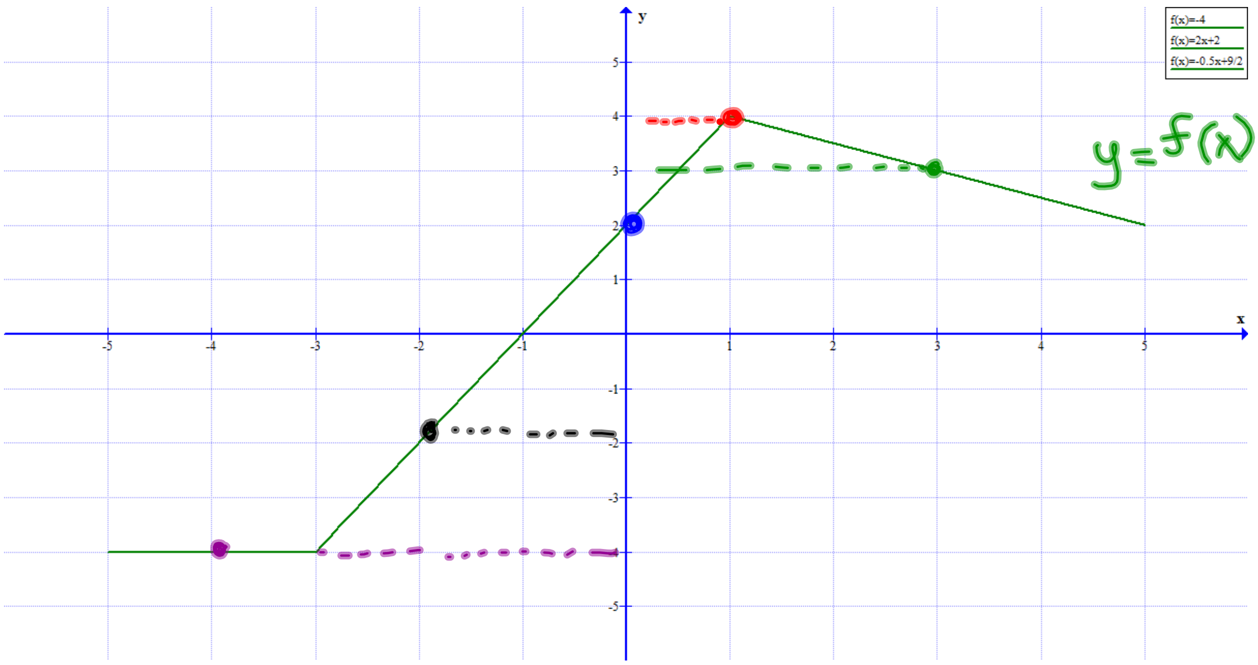
$(4x-5)(4x-5)$
$16x^2 - 20x - 20x + 25$

① $f(4x-5) = 3(4x-5)^2 + 2(4x-5) + 1$
$= 3(16x^2 - 40x + 25) + 8x - 10 + 1$
$= 48x^2 - 120x + 75 + 8x - 9$
$= 48x^2 - 112x + 66$ or $2(24x^2 - 56x + 33)$

② $g(3x^2+2x+1) = 4(3x^2+2x+1) - 5$
$= 12x^2 + 8x + 4 - 5$
$= 12x^2 + 8x - 1$

Given the graph of f(x) shown below, evaluate the following:

$$\frac{3f(1)-5\left[f(3)-7f(0)\right]}{2f(-2)-3f(-4)} = \frac{3(4)-5\left[3-7(2)\right]}{2(-2)-3(-4)} = \frac{12-5(-11)}{-4+12} = \frac{12+55}{8} = \boxed{\frac{67}{8}}$$



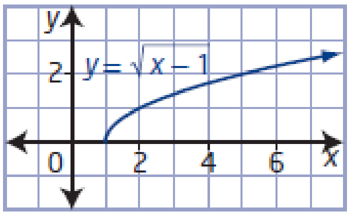
Key Ideas

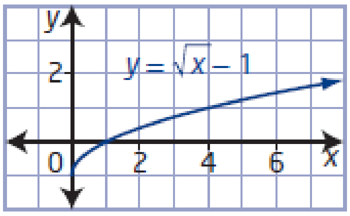
- Two functions, $f(x)$ and $g(x)$, can be combined using composition to produce two new functions, $f(g(x))$ and $g(f(x))$.
- To evaluate a composite function, $f(g(x))$, at a specific value, substitute the value into the equation for $g(x)$ and then substitute the result into $f(x)$ and evaluate, or determine the composite function first and then evaluate for the value of x.
- To determine the equation of a composite function, substitute the second function into the first as read from left to right. To compose $f(g(x))$, substitute the equation of $g(x)$ into the equation of $f(x)$.
- The domain of $f(g(x))$ is the set of all values of x in the domain of g for which $g(x)$ is in the domain of f. Restrictions on the inner function as well as the composite function must be considered.

Homework

#1-10 on page 507 (omit #6)

10.3 Composite Functions, pages 507 to 509

1. a) 3 b) 0 c) 2 d) −1
2. a) 2 b) 2 c) −4 d) −5
3. a) 10 b) −8 c) −2 d) 28
4. a) $f(g(a)) = 3a^2 + 1$ b) $g(f(a)) = 9a^2 + 24a + 15$
   c) $f(g(x)) = 3x^2 + 1$ d) $g(f(x)) = 9x^2 + 24x + 15$
   e) $f(f(x)) = 9x + 16$ f) $g(g(x)) = x^4 - 2x^2$
5. a) $f(g(x)) = x^4 + 2x^3 + 2x^2 + x$, $g(f(x)) = x^4 + 2x^3 + 2x^2 + x$
   b) $f(g(x)) = \sqrt{x^4 + 2}$, $g(f(x)) = x^2 + 2$
   c) $f(g(x)) = x^2$, $g(f(x)) = x^2$
6. a) 

   domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$
   b) 

   domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$, range $\{y \mid y \geq -1, y \in \mathrm{R}\}$
7. a) $g(x) = 2x - 5$ b) $g(x) = 5x + 1$
8. Christine is right. Ron forgot to replace all x's with the other function in the first step.
9. Yes. $k(j(x)) = j(k(x)) = x^6$; using the power law: $2(3) = 6$ and $3(2) = 6$.
10. No. $s(t(x)) = x^2 - 6x + 10$ and $t(s(x)) = x^2 - 2$.
11. a) $W(C(t)) = 3\sqrt{100 + 35t}$
    b) domain $\{t \mid t \geq 0, t \in \mathrm{R}\}$, range $\{W \mid W \geq 30, W \in \mathrm{W}\}$