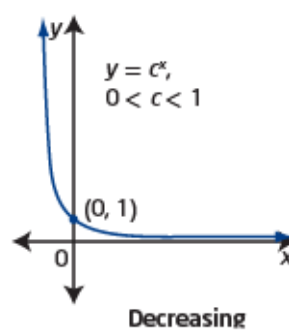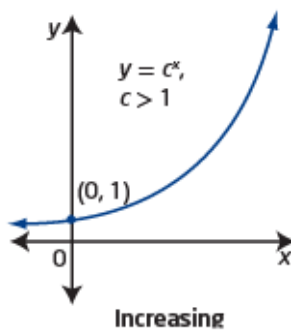# Exponential Functions

The graph of an **exponential function**, such as $y = c^x$, is increasing for $c > 1$, decreasing for $0 < c < 1$, and neither increasing nor decreasing for $c = 1$. From the graph, you can determine characteristics such as domain and range, any intercepts, and any asymptotes.

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable

Why is the definition of an exponential function restricted to positive values of c?

$y = c^x,$
$c > 1$

(0, 1)

Increasing

$y = c^x,$
$0 < c < 1$

(0, 1)

Decreasing

Did You Know?

Any letter can be used to represent the base in an exponential function. Some other common forms are $y = a^x$ and $y = b^x$. In this chapter, you will use the letter c. This is to avoid any confusion with the transformation parameters, $a, b, h,$ and $k,$ that you will apply in Section 7.2.

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in R\}$
  - has a range of $\{y \mid y > 0, y \in R\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$

increasing

$y = 2^x$

$(0, 1)$

decreasing

$y = \left(\frac{1}{2}\right)^x$

$(0, 1)$
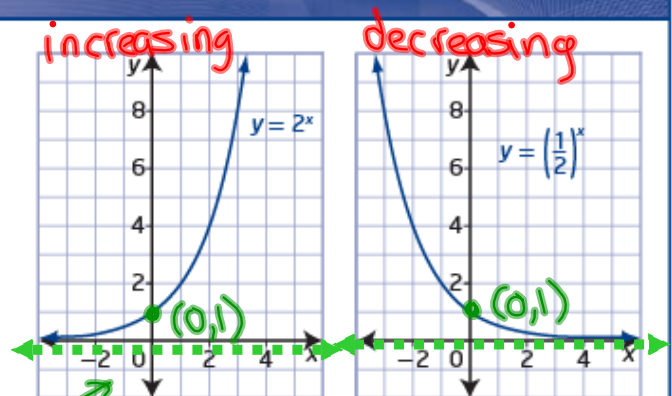
Example 1

Analyse the Graph of an Exponential Function

Graph each exponential function. Then identify the following:
- the domain and range
- the x-intercept and y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

a) $y = 4^x$

b) $f(x) = \left(\frac{1}{2}\right)^x$

a) $y = 4^x$

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y > 0, y \in R\}$

x-int: none

y-int: $y = 1$ or $(0,1)$

increasing $(c = 4)$

HA: $y = 0$

b) $f(x) = \left(\frac{1}{2}\right)^x$

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y > 0, y \in R\}$

x-int: none

y-int: $y = 1$ or $(0,1)$

decreasing $(c = \frac{1}{2})$

HA: $y = 0$

Solution

a) Method 1: Use Paper and Pencil

Use a table of values to graph the function.

Select integral values of x that make it easy to calculate the corresponding values of y for $y = 4^x$.

$y = 4^x$

x	y
−2	$\frac{1}{16}$
−1	$\frac{1}{4}$
0	1
1	4
2	16

Domain: $\{x | x \in R\}$

Range: $\{y | y > 0, y \in R\}$

x-int: none

y-int: $y = 1$ or $(0, 1)$

increasing $(c = 4)$

HA: $y = 0$

$y = 4^x$

$(2, 16)$

$(1, 4)$

$(0, 1)$

$\left(-2, \frac{1}{16}\right)$ $\left(-1, \frac{1}{4}\right)$
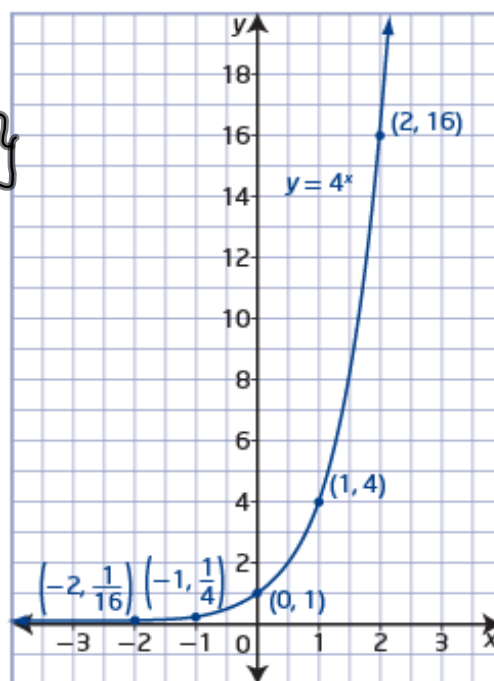
b) Method 1: Use Paper and Pencil

Use a table of values to graph the function.

Select integral values of x that make it easy to calculate the corresponding values of y for $f(x) = \left(\frac{1}{2}\right)^x$.

$y = \left(\frac{1}{2}\right)^x$

x	f(x)
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

Domain: $\{x | x \in R\}$

Range: $\{y | y > 0, y \in R\}$

x-int: none

y-int: $y = 1$ or $(0, 1)$

decreasing $(c = \frac{1}{2})$

HA: $y = 0$

$f(x) = \left(\frac{1}{2}\right)^x$

$(-3, 8)$
$(-2, 4)$
$(-1, 2)$
$(0, 1)$
$\left(1, \frac{1}{2}\right)$ $\left(2, \frac{1}{4}\right)$
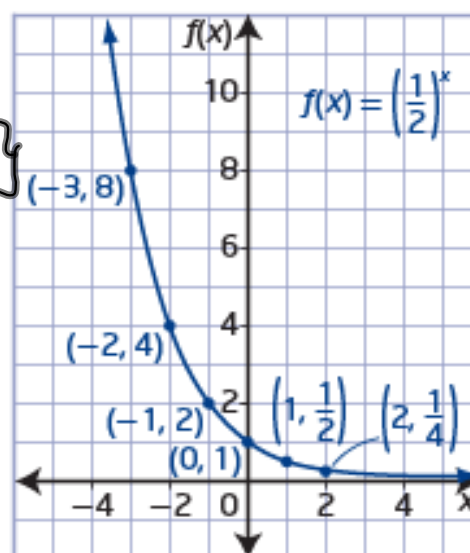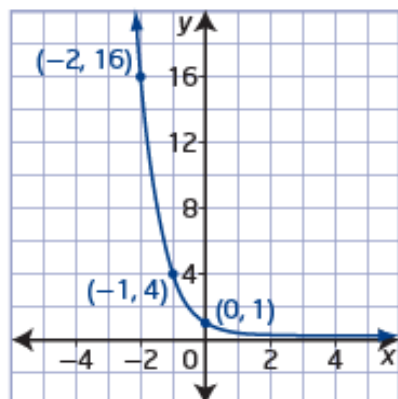
Example 2

Write the Exponential Function Given Its Graph

What function of the form $y = c^x$ can be used to describe the graph shown?

What is the value of c?

▸ Solution

Look for a pattern in the ordered pairs from the graph.

x	y
-2	16
-1	4
0	1

$\frac{1}{4}$

$\frac{1}{4}$

16, 4, 1 → form a geometric sequence.

As the value of x increases by 1 unit, the value of y decreases by a factor of $\frac{1}{4}$. Therefore, for this function, $c = \frac{1}{4}$.

Choose a point other than (0, 1) to substitute into the function $y = \left(\frac{1}{4}\right)^x$ to verify that the function is correct. Try the point (−2, 16).

Check:
Left Side   Right Side

Why should you not use the point (0, 1) to verify that the function is correct?

(0,1) is common to all exponential functions of the form $y = c^x$, $c > 0$

$y = \left(\frac{1}{4}\right)^x$

$16 \quad \left(\frac{1}{4}\right)^{-2}$

$\left(4\right)^2$

$16 \checkmark$

Example 3

Application of an Exponential Function

A radioactive sample of radium (Ra-225) has a **half-life** of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.

a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes?

b) What are the domain and range of this function?

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time, in 15-day intervals.

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.
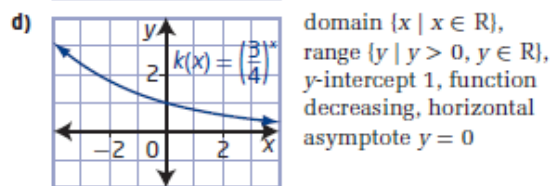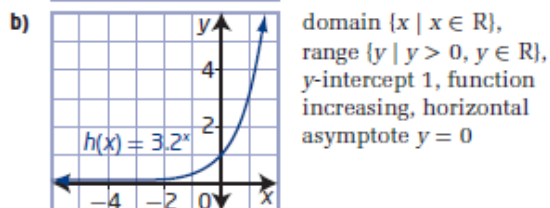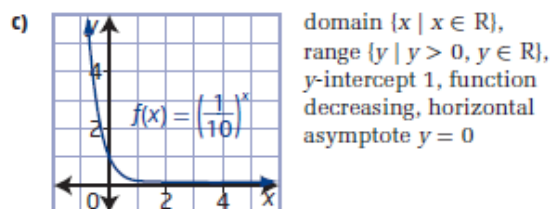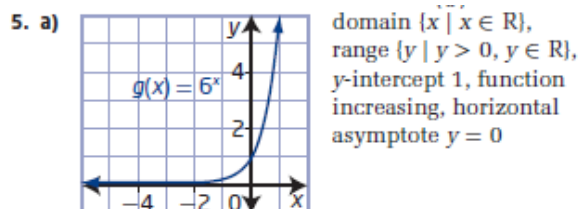
Mass of Ra-225 Remaining (g)

(1, 0.5)
(2, 0.25)
(3, 0.125)

Time (15-day intervals)
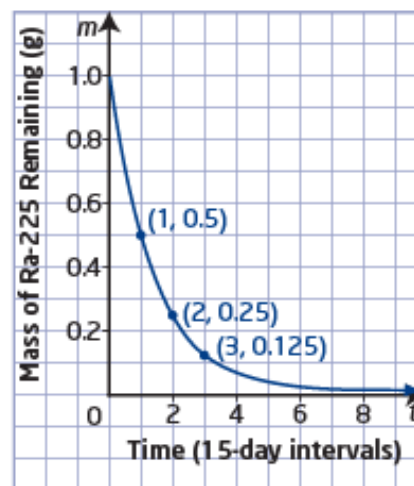
Homework

#1-8 on page 343

7.1 Characteristics of Exponential Functions, pages 342 to 345

1. a) No, the variable is not the exponent.
   b) Yes, the base is greater than 0 and the variable is the exponent.
   c) No, the variable is not the exponent.
   d) Yes, the base is greater than 0 and the variable is the exponent.

2. a) $f(x) = 4^x$ **b)** $g(x) = \left(\frac{1}{4}\right)^x$
   c) $x = 0$, which is the y-intercept

3. a) B   **b)** C   **c)** A

4. a) $f(x) = 3^x$ **b)** $f(x) = \left(\frac{1}{5}\right)^x$

5. a)

domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

b)

domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

c)

domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

d)

domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

**6. a)** $c > 1$; number of bacteria increases over time

   b) $0 < c < 1$; amount of actinium-225 decreases over time

   c) $0 < c < 1$; amount of light decreases with depth

   d) $c > 1$; number of insects increases over time

7. a)

The function $N = 2^t$ is exponential since the base is greater than zero and the variable t is an exponent.
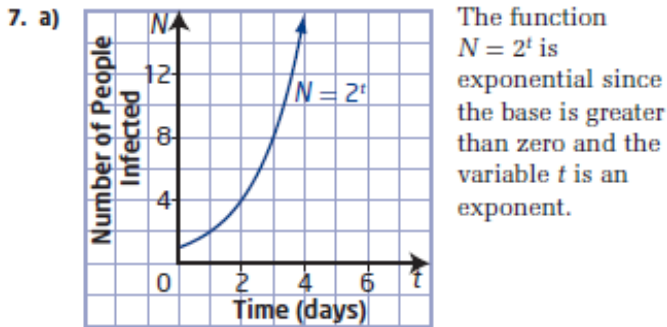
   b) **i)** 1 person   **ii)** 2 people
           iii) 16 people   **iv)** 1024 people

8. a) If the population increases by 10% each year, the population becomes 110% of the previous year's population. So, the growth rate is 110% or 1.1 written as a decimal.

   b)
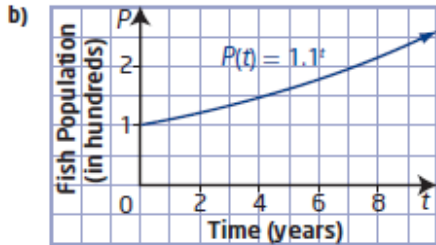
domain $\{t \mid t \geq 0, t \in R\}$ and range $\{P \mid P \geq 100, P \in R\}$

   c) The base of the exponent would become $100\% - 5\%$ or 95%, written as 0.95 in decimal form.

   d)
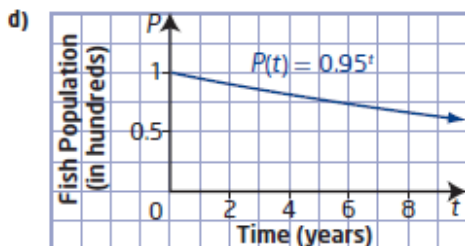
domain $\{t \mid t \geq 0, t \in R\}$ and range $\{P \mid 0 < P \leq 100, P \in R\}$