Example 3

Solve Problems Involving Exponential Equations With Different Bases

Christina plans to buy a car. She has saved $5000. The car she wants costs $5900. How long will Christina have to invest her money in a term deposit that pays 6.12% per year, compounded quarterly, before she has enough to buy the car?

Solution

The formula for compound interest is $A = P(1 + i)^n$, where A is the amount of money at the end of the investment; P is the principal amount deposited; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods.

6.12%
$= 0.0612 \div 4$
$= 0.0153$

In this problem:

$A = 5900$

$P = 5000$

$i = 0.0612 \div 4$ or 0.0153

Divide the interest rate by 4 because interest is paid quarterly or four times a year.

$$A = P(1+i)^n$$

$$\frac{5900}{5000} = \frac{\cancel{5000}(1.0153)^n}{\cancel{5000}}$$

$$1.18 = (1.0153)^n$$

$$(1.\cancel{0153})^{10.9} = (1.\cancel{0153})^n$$

* Express 1.18 with a base of 1.0153 → $\frac{\log 1.18}{\log 1.0153} = 10.9$

$$10.9 = n$$

n is the number of compound periods ... we want to know the number of years.

↳ $10.9 \div 4 = \boxed{2.73 \text{ years}}$

The number of milligrams of a drug remaining in the bloodstream t days after consumption is given by the equation:

$$D = 50(0.9)^t$$

(a) What percentage of the drug **leaves** the body each day? 10 %

(b) The drug can be detected in urine tests when 2 or more mg of the drug remain in the bloodstream. Will there be evidence of this drug in the bloodstream 28 days after consumption? Provide proof! $t=28$

$D=50(0.9)^{28}$

$D=2.62$ mg

Understanding Logarithms

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1.

Logarithmic Form **Exponential Form**

exponent

$\log_c x = y$ $c^y = x$

base

exp.

$2^4 = 16 \rightarrow \log_2 16 = 4$

base ans

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

a) $32 = 2^5$ exp.

ans. base

$\log_2(32) = 5$

or $5 = \log_2 32$

b) $2^{-5} = \dfrac{1}{32}$

$\log_2\left(\frac{1}{32}\right) = -5$

c) $x = 10^y$

$\log_{10} x = y$

or $y = \log x$

Logarithmic Form **Exponential Form**

exponent

$\log_c x = y$ $c^y = x$

base

Write each of the following in exponential form

a) $\log_4 16 = 2$ exp.

ans

base

$4^2 = 16$

b) $\log_2\left(\dfrac{1}{32}\right) = -5$

$2^{-5} = \frac{1}{32}$

c) $\log 65 = 1.8129$

$10^{1.8129} = 65$

Example 1

Evaluating a Logarithm

Evaluate.

a) $\log_7 49$

Let $x = \log_7 49$

$7^x = 49$ (exp. form)

$\cancel{7}^x = \cancel{7}^2$ (get common base)

$x = 2$

$\boxed{\log_7 49 = 2}$

b) $\log_6 1$

Let $x = \log_6 1$

$6^x = 1$

$6^x = 6^0$

$x = 0$

$\boxed{\log_6 1 = 0}$

c) $\log 0.001$

Let $x = \log 0.001$

$10^x = 0.001$

$10^x = 10^{-3}$

$x = -3$

$\boxed{\log 0.001 = -3}$

d) $\log_2 \sqrt{8}$

Let $x = \log_2 \sqrt{8}$

$2^x = \sqrt{8}$

$2^x = (2^3)^{\frac{1}{2}}$

$\cancel{2}^x = \cancel{2}^{3/2}$

$\boxed{x = \frac{3}{2}}$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

a) $5^{-3} = x$

$\left(\frac{1}{5}\right)^3 = x$

$\boxed{\frac{1}{125} = x}$

b) $x^2 = 36$

$x = \pm 6$

$\boxed{x = 6}$

c) $64^{2/3} = x$

$\boxed{16 = x}$

← (Inverse) →

Exponential Function	Logarithmic Function
$y = c^x$, $c > 0$, $c \neq 1$	$y = \log_c x$, $c > 0$, $c \neq 1$
D: $\{x \mid x \in R\}$	D: $\{x \mid x > 0, x \in R\}$
R: $\{y \mid y > 0, y \in R\}$	R: $\{y \mid y \in R\}$
HA: $y = 0$	VA: $x = 0$
x int: none	x int: $(1,0)$
y int: $(0,1)$	y int: none

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

a) Find Inverse:

$f(x) = 3^x$

① $y = 3^x$

② $\underset{\text{ans}}{x} = \underset{\text{Base}}{3}^{\overset{\text{exp}}{y}}$

③ $y = \log_3 x$ (Put in Logarithmic)

④ $f^{-1}(x) = \log_3 x$

$f(x) = 3^x$	$f^{-1}(x) = \log_3 x$
D: $\{x \mid x \in R\}$	D: $\{x \mid x > 0, x \in R\}$
R: $\{y \mid y > 0, y \in R\}$	R: $\{y \mid y \in R\}$
HA: $y = 0$	VA: $x = 0$
x int: none	x int: $(1, 0)$
y int: $(0, 1)$	y int: none

$y = 3^x \xleftrightarrow{\text{Inverse}} y = \log_3 x$

$(x, y) \rightarrow (y, x)$

$\boxed{\text{reflected in } y = x}$

$y = 3^x$

x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

$y = \log_3 x$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

Solution

a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $y = \log_3 x$. Since the inverse is a function, it can be written in function notation as $f^{-1}(x) = \log_3 x$.

$y = 3^x$ passes the HLT

How do you know that $y = \log_3 x$ is a function?

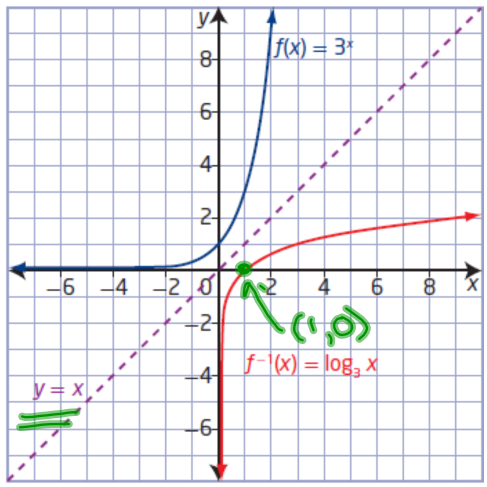
b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$f(x) = 3^x$

x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$

x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$
- the x-intercept is 1 or (1,0)
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

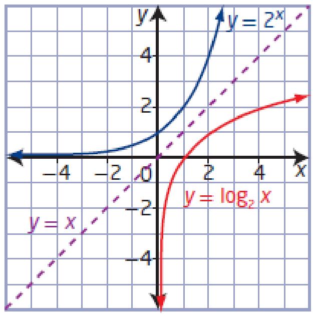
How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$
  - the range is $\{y \mid y \in \mathrm{R}\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

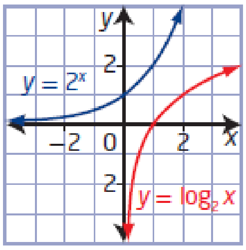
$$\log_{10} x = \log x$$

Homework

#1-5, 8, 10, 12, 13, 17 on page 380

8.1 Understanding Logarithms, pages 380 to 382

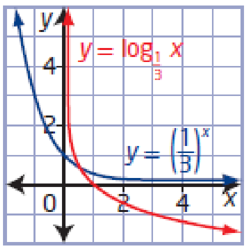
1. a) i)



ii) $y = \log_2 x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

b) i)



ii) $y = \log_{\frac{1}{3}} x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

2. a) $\log_{12} 144 = 2$ **b)** $\log_8 2 = \frac{1}{3}$

c) $\log_{10} 0.000\,01 = -5$ **d)** $\log_7 (y + 3) = 2x$

3. a) $5^2 = 25$ **b)** $8^{\frac{2}{3}} = 4$

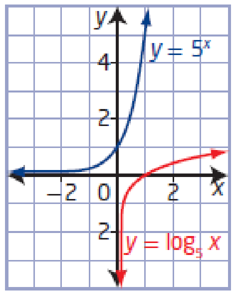
c) $10^6 = 1\,000\,000$ **d)** $11^y = x + 3$

4. a) 3 **b)** 0 **c)** $\frac{1}{3}$ **d)** -3

5. $a = 4$; $b = 5$

8. a) $y = \log_5 x$

b)



domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

10. They are reflections of each other in the line $y = x$.

11. a) They have the exact same shape.

b) One of them is increasing and the other is decreasing.

12. a) 216 **b)** 81 **c)** 64 **d)** 8

13. a) 7 **b)** 6

14. a) 0 **b)** 1

15. -1

16. 16

17. a) $t = \log_{1.1} N$ **b)** 145 days

18. The larger asteroid had a relative risk that was 1479 times as dangerous.

19. 1000 times as great

20. 5

21. $m = 14$, $n = 13$

22. $4n$

23. $y = 3^{2^x}$