Understanding Logarithms

Chapter 8 (page 370)

Focus on...

$(x,y) \rightarrow (y,x)$

reflection in the line $y=x$

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1. (Page 373)

Logarithmic Form | **Exponential Form**

exponent

$\log_c x = y$ | $c^y = x$

base

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

a) $32 = 2^5$ exp.

answer. base

$\boxed{\log_2(32) = 5}$

$5 = \log_2 32$

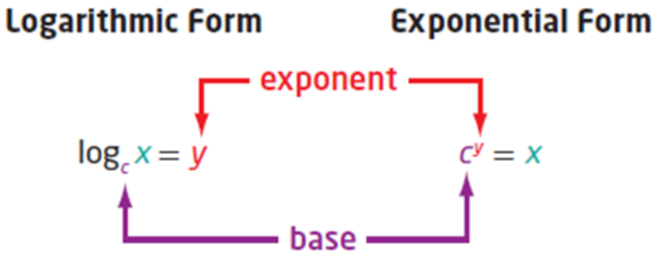
b) $2^{-5} = \dfrac{1}{32}$

$\boxed{\log_2\left(\frac{1}{32}\right) = -5}$

c) $x = 10^y$

$\log_{10} x = y$

$\boxed{\log x = y}$



Write each of the following in exponential form

a) $\log_4 16 = 2$ exp

base answer

$\boxed{4^2 = 16}$

$16 = 4^2$

b) $\log_2\left(\dfrac{1}{32}\right) = -5$

$\boxed{2^{-5} = \frac{1}{32}}$

c) $\log 65 = 1.8129$

$\boxed{10^{1.8129} = 65}$

Example 1

Evaluating a Logarithm

Evaluate. (Solving for an exponent)

a) $\log_7 49$ **b)** $\log_6 1$ **c)** $\log 0.001$ **d)** $\log_2 \sqrt{8}$

a) Let $x = \log_7 49$

$x = \log_7 49$

$7^x = 49$ ← express in exponential form →

$\cancel{7}^x = \cancel{7}^2$ ← get common base →

$x = 2$

$\therefore \log_7 49 = 2$

b) Let $x = \log_6 1$

$x = \log_6 1$

$6^x = 1$

$\cancel{6}^x = \cancel{6}^0$

$x = 0$

$\therefore \log_6 1 = 0$

c) $x = \log 0.001$

$10^x = 0.001$

$\cancel{10}^x = \cancel{10}^{-3}$

$x = -3$

$\therefore \log 0.001 = -3$

d) $x = \log_2 \sqrt{8}$

$2^x = \sqrt{8}$

$2^x = (8)^{1/2}$

$2^x = (2^3)^{1/2}$

$\cancel{2}^x = \cancel{2}^{3/2}$

$x = \frac{3}{2}$

$\therefore \log_2 \sqrt{8} = \frac{3}{2}$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$

b) $\log_x 36 = 2$

c) $\log_{64} x = \frac{2}{3}$

a) $\log_5 x = -3$

$5^{-3} = x$

$\left(\frac{1}{5}\right)^3 = x$

$\boxed{\frac{1}{125} = x}$

b) $\log_x 36 = 2$

$x^2 = 36$

$x = \pm 6$

$\boxed{\text{Choose } x = 6}$

c) $\log_{64} x = \frac{2}{3}$

$(64)^{2/3} = x$

$\boxed{16 = x}$

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

To Find Inverse:

a) $f(x) = 3^x$

$y = 3^x$ (Replace $f(x)$ with y)

$x = 3^y$ (Switch x + y)

$\boxed{y = \log_3 x}$ (Solve for y) → Express in logarithmic form

or $f^{-1}(x) = \log_3 x$

$f(x) = 3^x \longrightarrow f^{-1}(x) = \log_3 x$

$f(x) = 3^x$	$f^{-1}(x) = \log_3 x$
D: $\{x \mid x \in R\}$	D: $\{x \mid x > 0, x \in R\}$
R: $\{y \mid y > 0, y \in R\}$	R: $\{y \mid y \in R\}$
x-int: none	x-int: $(1,0)$
y-int: $(0,1)$	y-int: none
HA: $y=0$	VA: $x=0$

Solution

a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or, expressed in logarithmic form, $\boxed{y = \log_3 x.}$ Since the inverse is a function, it can be written in function notation as $\boxed{f^{-1}(x) = \log_3 x.}$

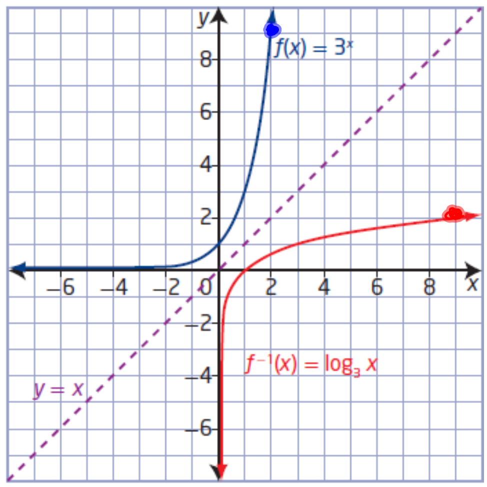
How do you know that $y = \log_3 x$ is a function?

$y=3^x$ passes the horizontal line test

b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in \mathrm{R}\}$ and the range is $\{y \mid y \in \mathrm{R}\}$
- the x-intercept is 1 (1,0)
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

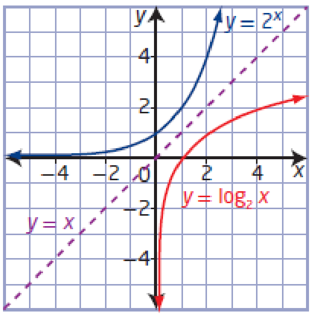
How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	Logarithmic Form
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in R\}$
  - the range is $\{y \mid y \in R\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

  $\log_{10} x = \log x$

Questions from Homework

8. a) If $f(x) = 5^x$, state the equation of the inverse, $f^{-1}(x)$.

b) Sketch the graph of $f(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

a) Inverse: $f(x) = 5^x$

$y = 5^x$

$x = 5^y$

$y = \log_5 x$

$\boxed{f^{-1}(x) = \log_5 x}$

For the curve $f(x) = 5^x$

D: $\{x \mid x \in R\}$

R: $\{y \mid y > 0, y \in R\}$

x-int: none

y-int: $(0,1)$

HA: $y = 0$

b) For the curve $f^{-1}(x) = \log_5 x$

D: $\{x \mid x > 0, x \in R\}$

R: $\{y \mid y \in R\}$

x-int: $(1,0)$

y-int: none

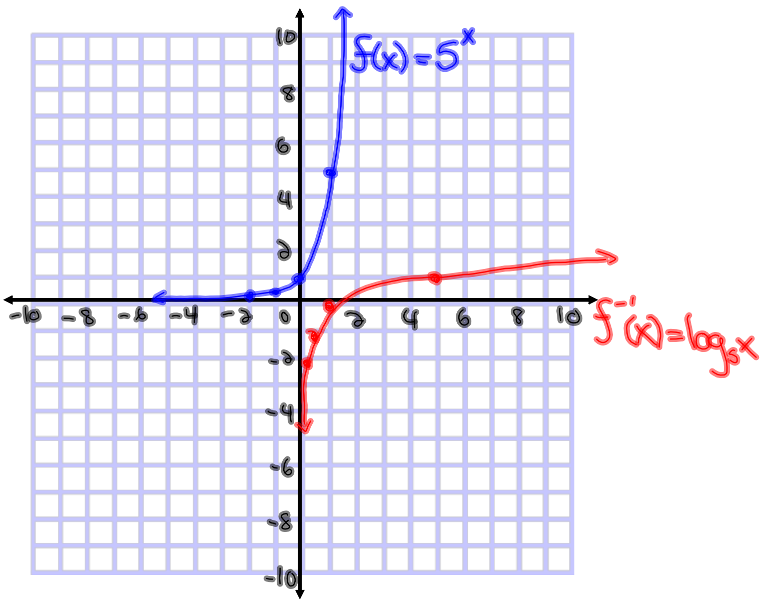
VA: $x = 0$

$f(x) = 5^x$

x	y
-2	$\frac{1}{25}$
-1	$\frac{1}{5}$
0	1
1	5
2	25

$f^{-1}(x) = \log_5 x$

x	y
$\frac{1}{25}$	-2
$\frac{1}{5}$	-1
1	0
5	1
25	2



17. The growth of a new social networking site can be modelled by the exponential function $N(t) = 1.1^t$, where N is the number of users after t days.

a) Write the equation of the inverse.

b) How long will it take, to the nearest day, for the number of users to exceed 1 000 000?

a) $N(t) = 1.1^t$

$f(x) = 1.1^x$

$y = 1.1^x$

$x = 1.1^y$

$y = \log_{1.1} x$

$f^{-1}(x) = \log_{1.1} x$

Transformations of Logarithmic Functions

Focus on...

- explaining the effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of $y = \log_c x$, where $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, where $c > 1$, and stating the characteristics of the graph

Remember:

Parameter	Transformation
a	$(x, y) \rightarrow (x, ay)$
b	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
h	$(x, y) \rightarrow (x + h, y)$
k	$(x, y) \rightarrow (x, y + k)$

Exponential Function ← (Inverse) →	Logarithmic Function
$y = c^x$, $c > 0$, $c \neq 1$	$y = \log_c x$, $c > 0$, $c \neq 1$
D: $\{x \mid x \in \mathbb{R}\}$	D: $\{x \mid x > 0, x \in \mathbb{R}\}$
R: $\{y \mid y > 0, y \in \mathbb{R}\}$	R: $\{y \mid y \in \mathbb{R}\}$
HA: $y = 0$	VA: $x = 0$
x int: none	x int: $(1,0)$
y int: $(0,1)$	y int: none

Example 1

Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = \log_3 (x + 9) + 2$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote
ii) the domain and range
iii) the y-intercept, if it exists
iv) the x-intercept, if it exists

a) $y = \log_3(x+9)+2$

$a=1$ $b=1$ $h=-9$ $k=2$

- Translated 9 to the left and 2 up
- Mapping: $(x,y) \longrightarrow (x-9, y+2)$

$f(x) = 3^x$

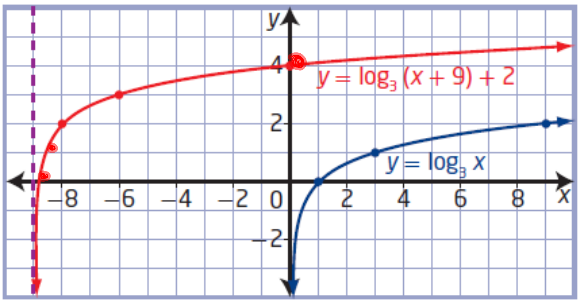
x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

$y = \log_3 x$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

$y = \log_3(x+9)+2$

x	y	
-80/9	0	x int
-26/3	1	
-8	2	
-6	3	
0	4	y int



b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote
ii) the domain and range
iii) the y-intercept, if it exists
iv) the x-intercept, if it exists

(i) VA: $x = -9$

(ii) D: $\{x \mid x > -9, x \in R\}$

R: $\{y \mid y \in R\}$

(iii) y-int: (Let $x=0$) $(0,4)$

(iv) x-int: (Let $y=0$) $\left(-\frac{80}{9}, 0\right)$

Example 2

Reflections, Stretches, and Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = -\log_2 (2x + 6)$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote
ii) the domain and range
iii) the y-intercept, if it exists
iv) the x-intercept, if it exists

a) $y = -\log_2(2x+6)$ ← factor

$y = -\log_2 2(x+3)$

$a=-1 \quad b=2 \quad h=-3 \quad k=0$

b) (i) VA: $x = -3$

(ii) D: $\{x \mid x > -3, x \in \mathbb{R}\}$

R: $\{y \mid y \in \mathbb{R}\}$

(iii) y-int: (Let $x=0$)

$y = -\log_2(2x+6)$

$y = -\log_2(2(0)+6)$

$y = -\boxed{\log_2 6}$

$\frac{\log 6}{\log 2}$ 2.58

$y = -2.58$

$(0, -2.58)$

(iv) x-int (Let $y=0$)

$y = -\log_2(2x+6)$

$0 = -\log_2(2x+6)$ Divide by -1

$0 = \log_2(2x+6)$ Change to Exponential Form

$2^0 = 2x+6$

$1 = 2x+6$

$-5 = 2x$

$-\frac{5}{2} = x$

$(-\frac{5}{2}, 0)$ or $(-2.5, 0)$

to sketch graph: Mapping Rule:

$(x,y) \to (\frac{1}{2}x - 3, -1y + 0)$

$f(x) = 2^x$

x	y
-2	1/4
-1	1/2
0	1
1	2
2	4

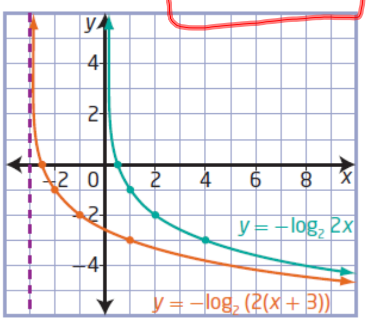
$y = \log_2 x$

x	y
1/4	-2
1/2	-1
1	0
2	1
4	2

Plot these points:

$y = -\log_2 2(x+3)$

x	y
-23/8	2
-11/4	1
-5/2	0
-2	-1
-1	-2



Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.

$$y = a \log_c (b(x - h)) + k$$

- a: Vertically stretch by a factor of $|a|$ about the x-axis. Reflect in the x-axis if $a < 0$.
- k: Vertically translate k units.
- b: Horizontally stretch by a factor of $\left|\frac{1}{b}\right|$ about the y-axis. Reflect in the y-axis if $b < 0$.
- h: Horizontally translate h units.

- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Homework

Questions #1, 2, 4, 5, 8, 11 on page 389 - 391