Introduction to Trigonometric Equations

trigonometric equation

- an equation involving trigonometric ratios

Focus on...

- algebraically solving first-degree and second-degree trigonometric equations in radians and in degrees
- verifying that a specific value is a solution to a trigonometric equation
- identifying exact and approximate solutions of a trigonometric equation in a restricted domain
- determining the general solution of a trigonometric equation

Did You Know?

In equations, mathematicians often use the notation $\cos^2 \theta$. This means the same as $(\cos \theta)^2$.

Let's start with basic LINEAR trigonometric equations...

...Pre-Calculus 110

Solve: $\sin\theta = 0.9659\ ,\ -360° < \theta < 720°$

- Reference angle?
- Which quadrants?
- Any co-terminal angles acceptable?

- If the domain is in degrees, give solutions in degrees.
- If the domain is in radians, give solutions in radians.

$\sin\theta = 0.9659\ ,\ -360° < \theta < 720°$

Where is sinθ positive

$\bar{\theta} = \sin^{-1}(0.9659)$

$\bar{\theta} = 75°$

reference angle

Q1	Q2
$\theta = \bar{\theta}$	$\theta = 180° - \bar{\theta}$
$\theta = 75°$	$\theta = 180° - 75°$
	$\theta = 105°$

To find angles between -360° and 720°

$75° - 360° = -285°$	$75° + 360° = 435°$
$105° - 360° = -255°$	$105° + 360° = 465°$

Solutions:

$\theta = -285°, -255°, 75°, 105°, 435°, 465°$

Solve: $\sec\theta = -1.3054\ ,\ -2\pi \le \theta \le 2\pi$ (radians)

(reciprocal)

$\cos\theta = \frac{1}{-1.3054}$

Where is $\cos\theta$ negative

$\cos\theta = -0.7660$

$\bar{\theta} = \cos^{-1}(0.7660)$

$\bar{\theta} = 0.698$

Q2	Q3
$\theta = \pi - \bar{\theta}$	$\theta = \pi + \bar{\theta}$
$\theta = 3.14 - 0.698$	$\theta = 3.14 + 0.698$
$\theta = 2.443$	$\theta = 3.838$

Find angles between -2π (-6.28) and 2π (6.28)

$2.443 - 6.28 = -3.838$

$3.838 - 6.28 = -2.443$

Solutions:

$\theta = -3.838, -2.443, 2.443, 3.838$

Exact Values → No Calculators

Ex. $\sqrt{2}\cos\theta + 1 = 0, \; -360° \le \theta \le 720°$

$\sqrt{2}\cos\theta = -1$

$\cos\theta = \frac{-1}{\sqrt{2}}$

Where is cosθ negative

Q2	Q3
$\theta = 180° - \bar{\theta}$	$\theta = 180° + \bar{\theta}$
$\theta = 180° - 45°$	$\theta = 180° + 45°$
$\theta = 135°$	$\theta = 225°$

(Use Special Triangles) $\bar{\theta} = 45°$

Find angles between -360° and 720°

$135° - 360° = -225°$	$135° + 360° = 495°$
$225° - 360° = -135°$	$225° + 360° = 585°$

Solutions.

$\theta = -225°, -135°, 135°, 225°, 495°, 585°$

Exact Value → No Calculator

Ex. $\sin x + 1 = 0, \; -2\pi \le x \le 4\pi$

$\sin x = -1$

(Unit Circle) $x = \frac{3\pi}{2}$

$\frac{3\pi}{2} - 2\pi$

$\frac{3\pi}{2} - \frac{4\pi}{2}$

$\frac{-\pi}{2}$

$\frac{3\pi}{2} + 2\pi$

$\frac{3\pi}{2} + \frac{4\pi}{2}$

$\frac{7\pi}{2}$

Solutions:

$x = \frac{-\pi}{2}, \frac{3\pi}{2}, \frac{7\pi}{2}$

Your Turn

Solve each trigonometric equation in the specified domain.

a) $3\cos\theta - 1 = \cos\theta + 1,\ -2\pi \leq \theta \leq 2\pi$

b) $4\sec x + 8 = 0,\ 0° \leq x < 360°$

Check Up:

radians

$\cot\theta = 0.7834 \qquad 0 < \theta < 2\pi$

$\tan\theta = 1.2765$

$\bar{\theta} = \tan^{-1}(1.2765)$

$\bar{\theta} = 0.9063$

Where is $\tan\theta$ positive

Q1	Q3
$\theta = \bar{\theta}$	$\theta = \pi + \bar{\theta}$
$\theta = 0.9063$	$\theta = 3.14 + 0.9063$
	$\theta = 4.0463$

Solutions:
$\theta = 0.9063, 4.0463$

Degrees

$2\cos\theta - 5 = -6 \qquad -360^0 < \theta < 360^0$

$2\cos\theta = -1$

$\cos\theta = -\frac{1}{2}$

$\bar{\theta} = 60^\circ$

Where is $\cos\theta$ negative?

Q2	Q3
$\theta = 180^\circ - \bar{\theta}$	$\theta = 180^\circ + \bar{\theta}$
$\theta = 180^\circ - 60^\circ$	$\theta = 180^\circ + 60^\circ$
$\theta = 120^\circ$	$\theta = 240^\circ$

Find angles between -360° and 360°

$120^\circ - 360^\circ = -240^\circ$

$240^\circ - 360^\circ = -120^\circ$

Solutions:
$\theta = -240^\circ, -120^\circ, 120^\circ, 240^\circ$

③ c) $5-\tan^2\theta = 4$, $-180° \le \theta \le 360°$

$-\tan^2\theta = -1$

$\tan^2\theta = 1$

$\tan\theta = \pm 1$

$\theta = 45°$

where is $\tan\theta$ (+/−)

Q1	Q2	Q3	Q4
$\theta = 45°$	$\theta = 135°$	$\theta = 225°$	$\theta = 315°$
~~$\theta = 315°$~~	~~$\theta = 225°$~~	$\theta = -135°$	$\theta = -45°$

Solutions:

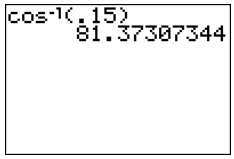
$\theta = -135°, -45°, 45°, 135°, 225°, 315°$

Questions from Homework

11) $\cos\theta = -0.15$

$\bar{\theta} = \cos^{-1}(0.15)$ when finding $\bar{\theta}$ always use positive value

$\bar{\theta} = 81.4°$



Where is $\cos\theta$ negative

Q2	Q3
$\theta = 180° - 81.4°$	$\theta = 180° + 81.4°$
$\theta = 98.6°$	$\theta = 261.4°$

$98.6° + 360°n, n \in I$

$261.4° + 360°n, n \in I$

Let's move onto QUADRATIC trigonometric equations...

...Pre-Calculus 110

(↑ (2nd Degree) (Factoring!!))

- What strategies can we use to solve quadratic equations?
- Quadratic trigonometric equations will ultimately become TWO linear trigonometric equations.

Solve: $2x^2 + x = 1$

$2x^2+x-1=0$

$(2x^2+2x)(-1x-1)=0$

$2x(x+1)-1(x+1)=0$

$(2x-1)(x+1)=0$

$2x-1=0$	$x+1=0$
$2x=1$	$x=-1$
$x=\frac{1}{2}$	

$2x-1=-2$

$2+-1=1$

Solve: $2\sin^2 x + \sin x = 1,\ 0 \le x \le 4\pi$

$2\sin^2x+1\sin x-1=0$

$(2\sin^2x+2\sin x)(-1\sin x-1)=0$

$2\sin x(\sin x+1)-1(\sin x+1)=0$

$(2\sin x-1)(\sin x+1)=0$

$2\sin x-1=0$	$\sin x+1=0$
$\sin x=\frac{1}{2}$	$\sin x=-1$
(Triangles) $\bar{x}=\frac{\pi}{6}$	(Unit Circle) $x=\frac{3\pi}{2}$

Q1	Q2
$x=\frac{\pi}{6}$	$x=\pi-\frac{\pi}{6}=\frac{5\pi}{6}$
$x=\frac{13\pi}{6}$	$x=\frac{17\pi}{6}$

$x=\frac{7\pi}{2}$

Solutions

$x=\frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6}, \frac{3\pi}{2}, \frac{7\pi}{2}$

Ex. $\cos^2\theta - \frac{1}{2}\cos\theta = 0, \; -2\pi \le \theta \le 4\pi$

$\cos\theta\left(\cos\theta - \frac{1}{2}\right) = 0$ (Common Factor)

$\cos\theta = 0$ (Unit Circle)

$\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

$\theta = \frac{5\pi}{2}, \frac{7\pi}{2}$

$\theta = -\frac{3\pi}{2}, -\frac{\pi}{2}$

$\cos\theta - \frac{1}{2} = 0$

$\cos\theta = \frac{1}{2}$ (Triangles)

$\theta = \frac{\pi}{3}$

Q1	Q4
$\theta = \frac{\pi}{3}$	$\theta = 2\pi - \frac{\pi}{3} = \frac{6\pi}{3} - \frac{\pi}{3} = \frac{5\pi}{3}$
$\theta = \frac{7\pi}{3}$	$\theta = \frac{11\pi}{3}$
$\theta = -\frac{5\pi}{3}$	$\theta = -\frac{\pi}{3}$

Ex. $6\sin^2 x - \sin x = 2, -2\pi \le \theta \le 4\pi$ (Decomposition)

General Solution of a Trigonometric Equation

$Solve: \quad 3\cos^2\theta - \cos\theta = 2; \ \theta \in \mathbb{R}$ (Decomposition)

Determine the general solution for $\sin^2 x - 1 = 0$ over the real numbers if x is measured in radians.

Did You Know?

$2n$, where $n \in \mathrm{I}$, represents all even integers.

$2n + 1$, where $n \in \mathrm{I}$, is an expression for all odd integers.

$\sin^2 x - 1 = 0$ (Difference of Squares)

$(\sin x + 1)(\sin x - 1) = 0$

$\sin x + 1 = 0$ | $\sin x - 1 = 0$

$\sin x = -1$ (unit circle) | $\sin x = 1$ (unit circle)

$\theta = \frac{3\pi}{2}$ | $\theta = \frac{\pi}{2}$

$x = \frac{\pi}{2} + 2\pi n$, where $n \in \mathrm{I}$

$x = \frac{3\pi}{2} + 2\pi n$, where $n \in \mathrm{I}$

or

$x = \frac{\pi}{2} + \pi n$, where $n \in \mathrm{I}$

or

$(2n + 1)\left(\frac{\pi}{2}\right)$, $n \in \mathrm{I}$

Determine the general solution for $\cos^2 x - 1 = 0$, where the domain is real numbers measured in degrees.

Questions from Homework

⑦ Solve $\sec^2\theta - 2\sec\theta - 3 = 0$ for $\theta \in [-180°, 180°)$

$(\sec\theta - 3)(\sec\theta + 1) = 0$

$-180° \le \theta < 180°$

$\sec\theta - 3 = 0$	$\sec\theta + 1 = 0$
$\sec\theta = 3$	$\sec\theta = -1$
$\cos\theta = \frac{1}{3}$	$\cos\theta = -1$
$\bar{\theta} = \cos^{-1}\left(\frac{1}{3}\right)$	(unit circle)
$\bar{\theta} = 70.5°$	$\theta = 180°$
	$\theta = -180°$

Q1	Q4
$\theta = 70.5°$	$\theta = 360° - 70.5$
$\theta = -289.5°$	$\theta = 289.5°$
	$\theta = -70.5°$

$\therefore \theta = -70.5°, 70.5°, -180°$

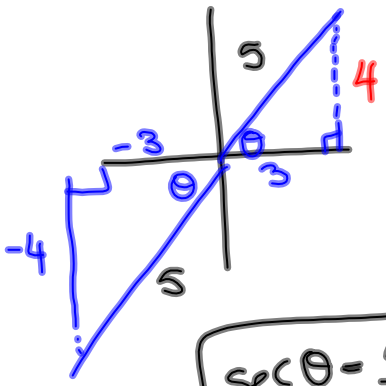
⑧ If $\cot\theta = \frac{3}{4}$, find the value of $\sec\theta$ ← $\frac{hyp}{adj}$

Given:

adj = 3 (x)

opp = 4 (y)

} Q1



$(3)^2 + (4)^2 = r^2$

$9 + 16 = r^2$

$25 = r^2$

$\pm 5 = r$

$\sec\theta = \frac{5}{3}$

$\sec\theta = -\frac{5}{3}$

Check-Up problem...

(Common Factor)

Solve: $\sin x \sec x + 2\sin x = 0 \ , \ x \in R$ (x is measured in radians)

$\sin x(\sec x + 2) = 0$

$\sin x = 0$

(Unit Circle)

$x = 0, \pi, 2\pi$

$0 + \pi n, n \in I$

$\sec x + 2 = 0$

$\sec x = -2$

$\cos x = -\frac{1}{2}$ (reciprocal)

(Triangle)

$\overline{x} = \frac{\pi}{3}$

Where is cosx negative

Q2	Q3
$x = \pi - \frac{\pi}{3} = \frac{2\pi}{3}$	$x = \pi + \frac{\pi}{3} = \frac{4\pi}{3}$
$\frac{2\pi}{3} + 2\pi n, n \in I$	$\frac{4\pi}{3} + 2\pi n, n \in I$

Unit Review...

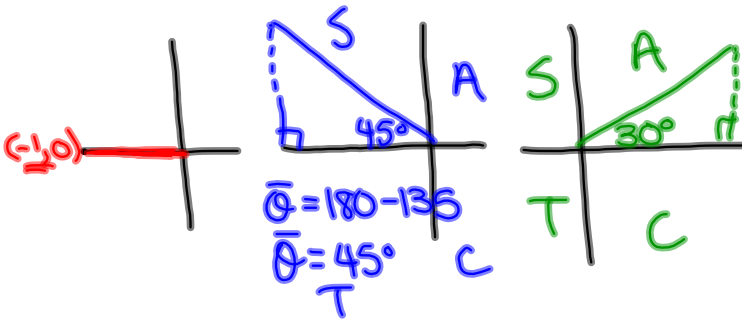
What topics have we covered??

- Radian Measure
- Co-terminal angles
- Principal Angles
- Angular Velocity (Open Response)
- The Unit Circle
- Trig Expressions (Open Response)
- Trig Equations (Open Response)

Review...

Without a calculator determine the value of...

$$\frac{2\cos 180^\circ + \sin 135^\circ}{\cos^2(-330^\circ)}$$



$$\frac{2(-1)+\left(\frac{1}{\sqrt{2}}\right)}{\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$4\cdot\frac{-2+\frac{\sqrt{2}}{2}}{\frac{3}{4}}\cdot 4$$

$$\frac{-8+2\sqrt{2}}{3}$$

Solve: $6\sin^2\theta - 3\sin\theta = 0$, $0 \le \theta \le 360°$

[A] $0°, 30°, 180°, 330°, 360°$

[B] $0°, 30°, 180°, 150°, 360°$

[C] $30°, 90°, 120°, 270°$

[D] $0°, 180°, 210°, 330°, 360°$

sinθ is negative + tanθ is positive (Quad 3)

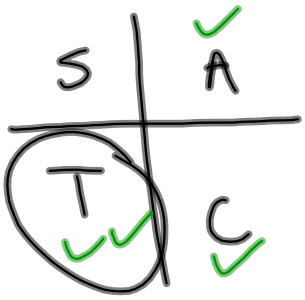
If $\csc\theta < 0$ and $\tan\theta > 0$, then which of the following could be a possible measure of angle θ?

[A] $\frac{11\pi}{6}$ (Q4)

[B] $\frac{4\pi}{3}$ (Q3)

[C] $\frac{3\pi}{4}$ (Q2)

[D] $\frac{\pi}{2}$



$-\frac{25\pi}{4} + \frac{8\pi}{1} = -\frac{25\pi}{4} + \frac{32\pi}{4} = \frac{7\pi}{4}$

What is the principal angle of $-\frac{25\pi}{4}$?

POSITIVE

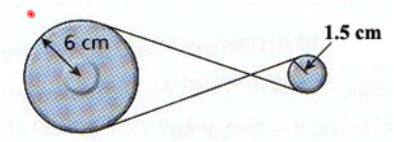
[A] $\frac{3\pi}{4}$

[B] $\frac{\pi}{4}$

[C] $-\frac{\pi}{4}$

[D] $\frac{7\pi}{4}$

If the belt in the pulley system below travels 30 cm, what is the angle of rotation of the smaller pulley?



[A] $\frac{\pi}{9}$ radians [B] 20°

[C] 20 radians [D] 5°

small

$r = 1.5\text{ cm}$

$a = 30\text{cm}$

$\theta = ?$

$$\theta = \frac{a}{r} = \frac{30\text{cm}}{1.5\text{cm}} = 20\text{rads}$$

Nibbles the hamster is running at 0.02 m/s on an exercise wheel of radius 8 cm. What is the angular velocity of this wheel?

[A] 0.15 rad/minute [B] 240 rad/minute [C] 0.25 rad/minute [D] 15 radians/minute

Given:

$r = 8\text{cm}$

$a = 2\text{cm/s} \times 1\text{s} = 2\text{cm}$

① Find θ

$$\theta = \frac{a}{r} = \frac{2}{8} = 0.25\text{rads}$$

② Find V_a

$$V_a = \frac{\theta}{t} = \frac{0.25\text{rads}}{1\text{sec}} \times \frac{60\text{sec}}{1\text{min}} = \frac{15\text{rads}}{\text{min}}$$

Solve: $2(1-\sin\theta)^2+\sin\theta=2(3-4\sin^2\theta)\,,\ -360°\le\theta\le 720°$

Chapter 4 Review:

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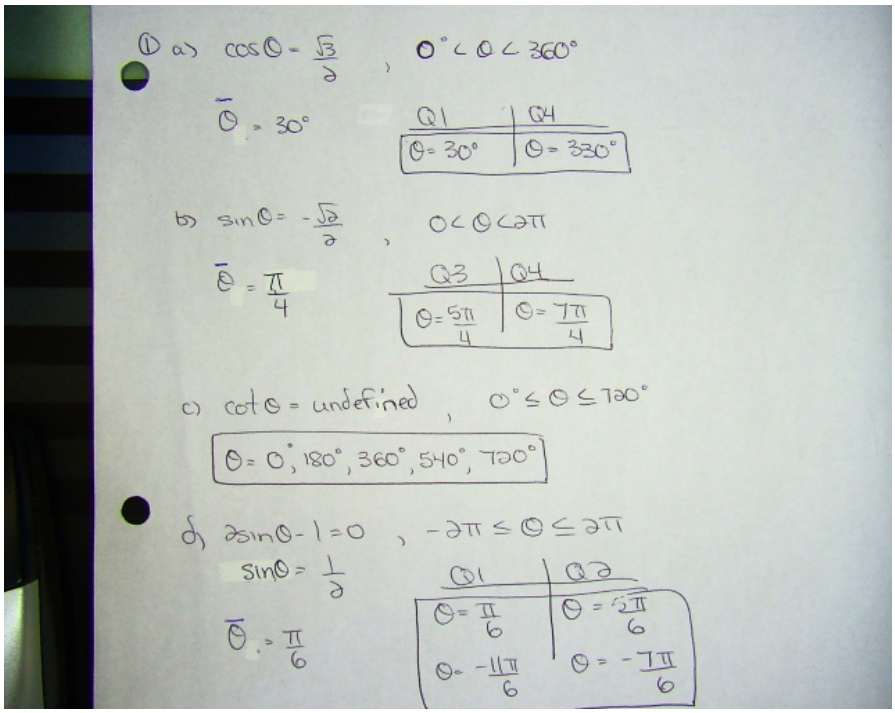
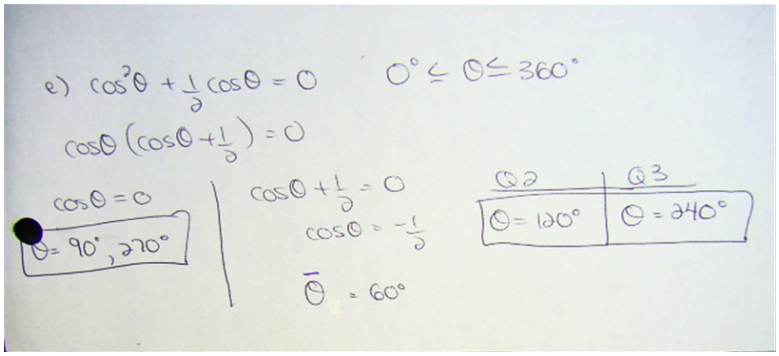
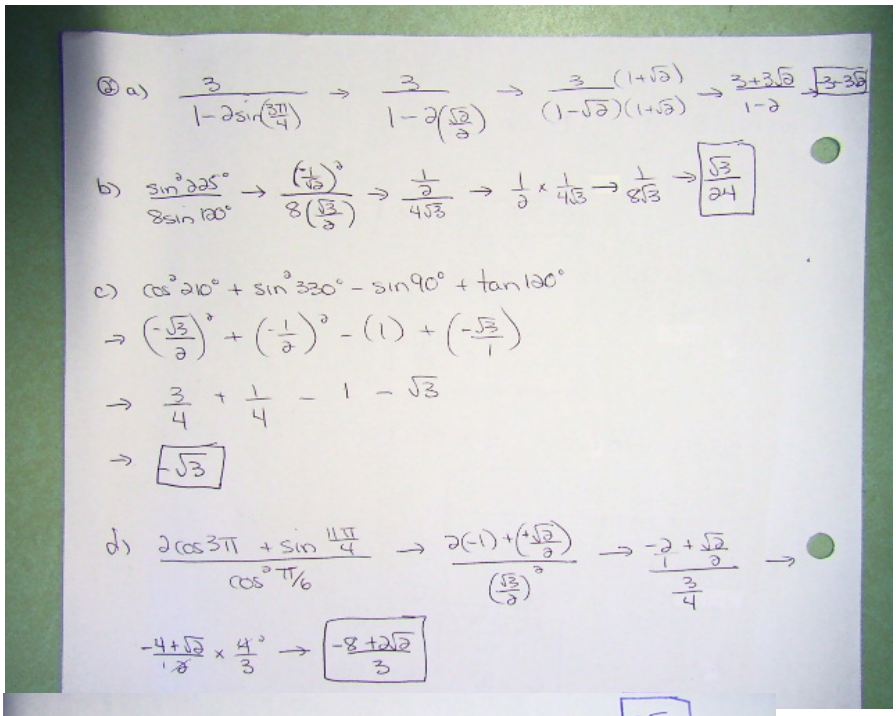
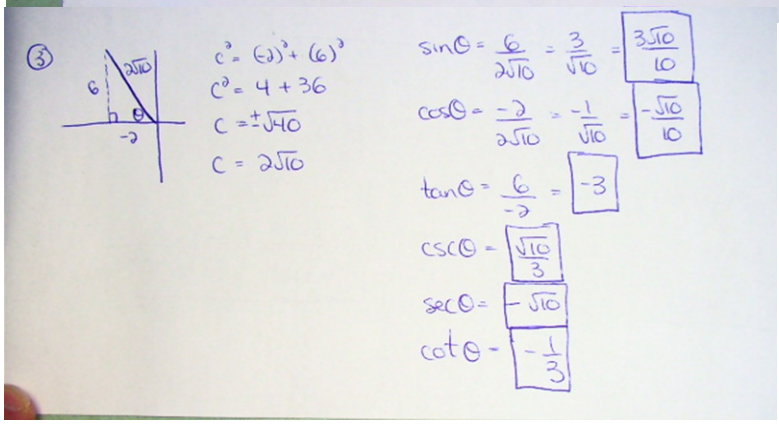
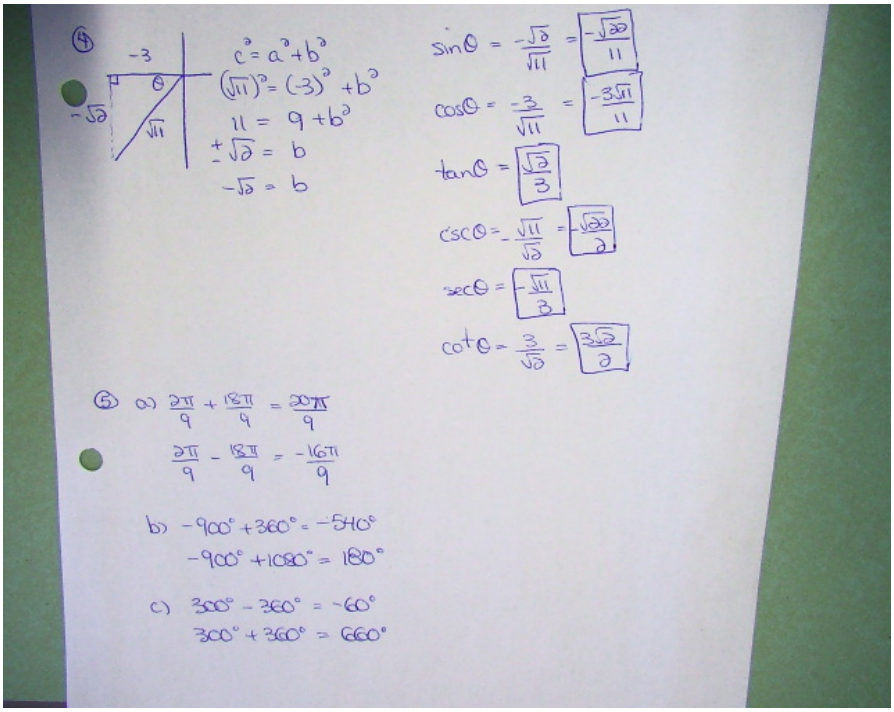

① a) cosθ = √3/2 , 0° < θ < 360°
θ̄ = 30°
Q1 | Q4
θ = 30° | θ = 330°
b) sinθ = -√2/2 , 0 < θ < 2π
θ̄ = π/4
Q3 | Q4
θ = 5π/4 | θ = 7π/4
c) cotθ = undefined , 0° ≤ θ ≤ 720°
θ = 0°, 180°, 360°, 540°, 720°
d) 2sinθ - 1 = 0 , -2π ≤ θ ≤ 2π
sinθ = 1/2
θ̄ = π/6
Q1 | Q2
θ = π/6 | θ = 5π/6
θ = -11π/6 | θ = -7π/6


e) cos²θ + 1/2 cosθ = 0 0° ≤ θ ≤ 360°
cosθ(cosθ + 1/2) = 0
cosθ = 0
θ = 90°, 270°
cosθ + 1/2 = 0
cosθ = -1/2
θ̄ = 60°
Q2 | Q3
θ = 120° | θ = 240°


② a) $\frac{3}{1-2\sin(\frac{3\pi}{4})} \rightarrow \frac{3}{1-2(\frac{\sqrt{2}}{2})} \rightarrow \frac{3}{(1-\sqrt{2})(1+\sqrt{2})}(1+\sqrt{2}) \rightarrow \frac{3+3\sqrt{2}}{1-2}$ $-3-3\sqrt{2}$
b) $\frac{\sin^2 225^\circ}{8\sin 120^\circ} \rightarrow \frac{(-\frac{1}{\sqrt{2}})^2}{8(\frac{\sqrt{3}}{2})} \rightarrow \frac{\frac{1}{2}}{4\sqrt{3}} \rightarrow \frac{1}{2} \times \frac{1}{4\sqrt{3}} \rightarrow \frac{1}{8\sqrt{3}} \rightarrow \frac{\sqrt{3}}{24}$
c) $\cos^2 210^\circ + \sin^2 330^\circ - \sin 90^\circ + \tan 180^\circ$
$\rightarrow (-\frac{\sqrt{3}}{2})^2 + (-\frac{1}{2})^2 - (1) + (-\frac{\sqrt{3}}{1})$
$\rightarrow \frac{3}{4} + \frac{1}{4} - 1 - \sqrt{3}$
$\rightarrow -\sqrt{3}$
d) $\frac{2\cos 3\pi + \sin\frac{11\pi}{4}}{\cos^2 \pi/6} \rightarrow \frac{2(-1)+(+\frac{\sqrt{2}}{2})}{(\frac{\sqrt{3}}{2})^2} \rightarrow \frac{-2+\frac{\sqrt{2}}{2}}{\frac{3}{4}} \rightarrow$
$\frac{-4+\sqrt{2}}{2} \times \frac{4}{3} \rightarrow \frac{-8+2\sqrt{2}}{3}$


③
$2\sqrt{10}$
6
θ
-2
$c^2 = (-2)^2 + (6)^2$
$c^2 = 4 + 36$
$c = \pm\sqrt{40}$
$c = 2\sqrt{10}$
$\sin\theta = \frac{6}{2\sqrt{10}} = \frac{3}{\sqrt{10}} = \frac{3\sqrt{10}}{10}$
$\cos\theta = \frac{-2}{2\sqrt{10}} = \frac{-1}{\sqrt{10}} = \frac{-\sqrt{10}}{10}$
$\tan\theta = \frac{6}{-2} = -3$
$\csc\theta = \frac{\sqrt{10}}{3}$
$\sec\theta = -\sqrt{10}$
$\cot\theta = -\frac{1}{3}$



④

$c^2 = a^2 + b^2$

$(\sqrt{11})^2 = (-3)^2 + b^2$

$11 = 9 + b^2$

$\pm\sqrt{2} = b$

$-\sqrt{2} = b$

$\sin\theta = \frac{-\sqrt{5}}{\sqrt{11}} = \boxed{\frac{-\sqrt{55}}{11}}$

$\cos\theta = \frac{-3}{\sqrt{11}} = \boxed{\frac{-3\sqrt{11}}{11}}$

$\tan\theta = \boxed{\frac{\sqrt{5}}{3}}$

$\csc\theta = \frac{\sqrt{11}}{\sqrt{5}} = \boxed{\frac{\sqrt{55}}{5}}$

$\sec\theta = \boxed{-\frac{\sqrt{11}}{3}}$

$\cot\theta = \frac{3}{\sqrt{5}} = \boxed{\frac{3\sqrt{5}}{5}}$

⑤ a) $\frac{2\pi}{9} + \frac{18\pi}{9} = \frac{20\pi}{9}$

$\frac{2\pi}{9} - \frac{18\pi}{9} = -\frac{16\pi}{9}$

b) $-900° + 360° = -540°$

$-900° + 1080° = 180°$

c) $300° - 360° = -60°$

$300° + 360° = 660°$

Little Johnny has a rock tied to the end of a piece of rope 1.5 m long and he is swinging it around his head in a circular pattern. Mrs. Centripetal, his physics teacher, is watching Johnny out the window of her physics lab and notes that the rock is making 12 revolutions every 48 seconds.

(a) Determine the angular velocity with which little Johnny is twirling the rope above his head. [2]

(b) The rock comes flying from the rope 3 minutes after Mrs. Centripetal started to time little Johnny. How far did the rock travel during the 3 minutes? [2]

Attachments

Worksheet - Sketching Angles in Radians.doc