Differential and Integral Calculus 120

$\int$ Substitution Rule! $\int$

Substitution Rule

It often arises that we need to integrate a function which does not follow one of our basic integral formulas therefore other techniques are necessary.

One such technique is called Substitution that involves a change of variable, which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$

Substitution Rule for Integration corresponds... to the Chain Rule for Differentiation.

Let's do an example...

Find $\int (x^2 - 5)^8 2x dx$

In using the substitution rule, the idea is to replace a complicated integral by a simpler integral by changing to a new variable u.

In thinking of the appropriate substitution, we try to choose u to be some function in the integrand whose differential du also occurs. (ignoring any constants)

So using the above example we have...

$$\int (x^2 - 5)^8 2x dx$$

Let:

$u = x^2 - 5$

$du = 2x dx$

$$\int (u)^8 du$$

$$= \frac{u^9}{9} + C$$

$$= \frac{(x^2-5)^9}{9} + C$$

Evaluate each of the following:

$$\int \frac{x^2}{\sqrt{1-x^3}}dx$$

$u = 1-x^3$

$du = -3x^2 dx$

$-\frac{1}{3}du = x^2 dx$

$$= \int \frac{x^2}{(1-x^3)^{1/2}} dx$$

$$= \int (u)^{-1/2} \cdot \left(-\frac{1}{3} du\right)$$

$$= -\frac{1}{3}\int (u)^{-1/2} du$$

$$= -\frac{1}{3} \cdot 2u^{1/2} + C$$

$$= -\frac{2}{3}(1-x^3)^{1/2} + C$$

$$= -\frac{2}{3}(1-x^3)^{1/2} + C$$

$$\int \sin 4x dx$$

$u = 4x$

$du = 4dx$

$\frac{1}{4}du = dx$

$$= \int \sin u \cdot \left(\frac{1}{4} du\right)$$

$$= \frac{1}{4}\int \sin(u)\, du$$

$$= \frac{1}{4} \cdot -\cos u + C$$

$$= \frac{-1}{4}\cos u + C$$

$$= -\frac{1}{4}\cos 4x + C$$

$$= -\frac{1}{4}\cos 4x + C$$

Evaluate each of the following:

$$\int \frac{\ln x}{x}dx = \int \ln x \cdot \frac{1}{x}dx$$

$u = \ln x$

$du = \frac{1}{x}dx$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln x)^2}{2} + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

$$\int (2+\sin x)^{10}\cos x\,dx$$

$u = 2+\sin x$

$du = \cos x\,dx$

$$= \int u^{10}\,du$$

$$= \frac{u^{11}}{11} + C$$

$$= \frac{(2+\sin x)^{11}}{11} + C$$

$$= \frac{1}{11}(2+\sin x)^{11} + C$$

Evaluate each of the following:

$$\int 2x\sqrt{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$$= \int u^{1/2}\,du$$

$$= \frac{2}{3}u^{3/2} + C$$

$$= \frac{2}{3}(1+x^2)^{3/2} + C$$

$$= \frac{2}{3}(x^2+1)^{3/2} + C$$

$$\int x^3\cos(x^4+2)\,dx$$

$u = x^4+2$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$$= \int \cos(u)\cdot\frac{1}{4}\,du$$

$$= \frac{1}{4}\int \cos(u)\,du$$

$$= \frac{1}{4}\sin u + C$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

$$= \frac{1}{4}\sin(x^4+2) + C$$

Evaluate each of the following:

$$\int \sqrt{2x+1}\,dx$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \int u^{1/2} \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{2u^{3/2}}{3} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$u = 1-4x^2$

$du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$$= \int u^{-1/2} \cdot \left(-\frac{1}{8}du\right)$$

$$= -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8} \cdot 2u^{1/2} + C$$

$$= -\frac{1}{4}u^{1/2} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Evaluate the following:

$$\int e^{5x}\,dx$$

$u = 5x$

$du = 5dx$

$\frac{1}{5}du = dx$

$$= \int e^{u} \cdot \left(\frac{1}{5}du\right)$$

$$= \frac{1}{5}\int e^{u}\,du$$

$$= \frac{1}{5}e^{u} + C$$

$$= \frac{1}{5}e^{5x} + C$$

$$= \frac{1}{5}e^{5x} + C$$

Evaluate: $\int \sqrt{1+x^2}\, x^5\, dx$

appropriate substitution becomes more obvious if we factor x^5 as x^4 x

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x^2 = u-1$

$x^4 = (u-1)^2$

$$\int \sqrt{1+x^2} \cdot x^4 \cdot x\,dx$$

$$= \int u^{1/2} \cdot (u-1)^2 \cdot \left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int u^{1/2} \cdot (u-1)^2\, du$$

$$= \frac{1}{2}\int u^{1/2}\left(u^2 - 2u + 1\right) du$$

$$= \frac{1}{2}\int u^{5/2} - 2u^{3/2} + u^{1/2}\, du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2} - \frac{2\cdot 2}{5}u^{5/2} + \frac{2}{3}u^{3/2}\right) + C$$

$$= \frac{1}{7}u^{7/2} - \frac{2}{5}u^{5/2} + \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{7}\left(1+x^2\right)^{7/2} - \frac{2}{5}\left(1+x^2\right)^{5/2} + \frac{1}{3}\left(1+x^2\right)^{3/2} + C$$

Homework - Exercise 11.3 - pp. 511-512 - Q. 1 - 3

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