Differential and Integral Calculus 120

Substitution Rule!

Substitution Rule

It often arises that we need to integrate a function which does not follow one of our basic integral formulas therefore other techniques are necessary.

One such technique is called <u>Substitution</u> that involves a change of variable, which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Substitution Rule for Indefinite Integrals

If
$$u = g(x)$$
 , then $\int f(g(x))g'(x)dx = \int f(u)du$

Substitution Rule for Integration corresponds...

Let's do an example... to the Chain Rule for Differentiation.

Find
$$\int (x^2 - 5)^8 2x dx$$

In using the substitution rule, the idea is to replace a complicated integral by a simpler integral by changing to a new variable u.

In thinking of the appropriate substitution, we try to choose u to be some function in the integrand whose differential du also occurs. (ignoring any constants)

So using the above example we have...

Let:

$$u = x^{2} - 5$$

$$du = 2xdx$$

$$\int (u)^{8} du$$

$$= \frac{u}{9} + C$$

$$= \frac{(x^{2} - 5)^{9} + C}{9}$$

$$\int \frac{x^{2}}{\sqrt{1-x^{3}}} dx \qquad \int \sin 4x dx$$

$$u = 1-x^{3} = \int \frac{x^{3}}{(1-x^{3})^{3}} dx \qquad u = 4x = \int \sin u \cdot (\frac{1}{4}du)$$

$$du = -3x^{3}dx = \int (\frac{1}{1-x^{3}})^{3}dx \qquad du = 4dx = \frac{1}{4}\int \sin(u) du$$

$$-\frac{1}{3}du = x^{3}dx = \int (\frac{1}{3}du) \frac{1}{4}du = dx = \frac{1}{4}\int \sin(u) du$$

$$= -\frac{1}{3}\int (\frac{1}{3}du) \frac{1}{3}du = \frac{1}{4}\int \cos(u) du$$

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$$= -\frac{1}{4}\int \cos(u) du = \frac{1}{4}\int \cos(u) du$$

$$= -\frac{1}{4}\cos(u) + C$$

$$= -\frac{1}{4}\cos(4x + C)$$

$$= -\frac{1}{4}\cos(4x + C)$$

$$\int \frac{\ln x}{x} dx = \int \ln x \cdot \int dx$$

$$u = \ln x$$

$$u = \ln x$$

$$= \int u du$$

$$du = \int dx$$

$$= \int u du$$

$$= \int u$$

$$\int 2x\sqrt{1+x^2} dx$$

$$\int x^3 \cos(x^4+2) dx$$

$$\int x^3$$

$$\int \sqrt{2x+1} \, dx$$

$$\frac{1}{\sqrt{1-4x^2}} \, dx$$

$$= \int u^{1/3} \, dx$$

$$= \int u^{1/3} \, dx$$

$$= -\frac{1}{8} \int u^{1/3} \, dx$$

Evaluate the following:

$$\int e^{5x} dx$$

$$= \int e^{4} (\int du)$$

$$du = 5dx$$

$$= \int \int e^{4} du$$

$$= \int \int e^{5} dx$$

Evaluate:

$$\int \sqrt{1+x^2} x^5 dx$$

appropriate substitution becomes more obvious if we factor x^5 as x^4 x

$$\frac{u = 1 + x^{3}}{du = 3x dx}$$

$$= \int u^{1/3} \cdot (u - 1)^{3} \cdot (\frac{1}{2} du)$$

$$= \frac{1}{3} \int u^{1/3} \cdot (u - 1)^{3} du$$

$$= \frac{1}{3} \int u^{1/3} \cdot (\frac{1}{2} - \frac{3}{2} u + 1) du$$

$$= \frac{1}{3} \int u^{1/3} \cdot (\frac{3}{2} - \frac{3}{2} u + \frac{3}{2} u^{1/3}) + C$$

$$= \frac{1}{7} (1 + x^{2})^{7/2} - \frac{2}{5} (1 + x^{2})^{5/2} + \frac{1}{3} (1 + x^{2})^{3/2} + C$$

<u>Homework</u> - Exercise 11.3 - pp. 511-512 - Q. 1 - 3 **RED BOOK**

Let's do one where the higher degree is not under the radical, but outside... how do we handle this????

$$\int x^2 \sqrt{8x + 5} dx$$

$$=\frac{(8x+5)^{\frac{3}{2}}}{672} \left[24x^2 - 12x + 5\right] + C$$

So,
$$\int x^2 \sqrt{8x+5} dx$$
 well let $u = \sqrt{8x+5}$ therefore $u^2 = 8x+5$ and $u = \sqrt{8x+5}$

$$2udu = 8dx$$

$$\frac{1}{4}udu = dx$$

$$= \int x^2 \sqrt{8x + 5} dx$$

$$= \int \left(\frac{u^2 - 5}{8}\right)^2 u \left(\frac{1}{4}u du\right)$$

$$= \int \left(\frac{u^4 - 10u^2 + 25}{64}\right) \left(\frac{u^2}{4} du\right)$$

$$= \int \frac{u^6 - 10u^4 + 25u^2}{256} du$$

$$= \frac{1}{256} \int u^6 - 10u^4 + 25u^2 du$$

$$=\frac{1}{256}\left(\frac{u^7}{7}-2u^5+\frac{25u^3}{3}\right)_{+c}$$

So, now fill in for u and simplify.....

$$=\frac{1}{256}\left(\frac{(8x+5)^{\frac{7}{2}}}{7}-2(8x+5)^{\frac{5}{2}}+\frac{25(8x+5)^{\frac{3}{2}}}{3}\right)+C$$

a 1/21 as well as (8x+5)^3/2

$$= \frac{(8x+5)^{\frac{3}{2}}}{5376} \left[3(8x+5)^2 - 42(8x+5) + 175 \right] + C$$

mulply this out

maybe now factor out

$$= \frac{(8x+5)^{\frac{3}{2}}}{5376} \left[192x^2 - 96x + 40 \right] + C$$
reduce

and then you can

$$=\frac{(8x+5)^{\frac{3}{2}}}{672} \left[24x^2-12x+5\right]+C$$

amazing!

Questions From Homework

$$= \frac{2}{1} \ln |u| + C$$

When performing a definite integration using substitution, we have to change the limits of integration so that they are the appropriate values of u.

Substitution Rule for Definite Integrals

If
$$u = g(x)$$
, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

Example...

$$\int_{1}^{9} \frac{e^{\sqrt{x}}}{\sqrt{x}} dx = \int_{1}^{3} e^{x} dx$$

$$= 2 \int_{1}^{3} e^{x} dx$$

$$dx = \frac{1}{2} dx$$

$$= 2 e^{x} - 2e^{x}$$

Find the area under the curve....

$$A = \int_0^1 \frac{1}{2x+1} dx = \int_0^3 \frac{1}{u} \cdot (\frac{1}{2} du)$$

$$u = 2x+1$$

$$du = 2dx$$

$$= \frac{1}{2} \int_0^3 \frac{1}{u} du$$

$$= \frac{1}{2} \ln |u| \int_0^3 \frac{1}{u} du$$

Evaluate:
$$= \frac{26}{3}$$

$$\int_{0}^{4} \sqrt{2x+1} dx$$

$$u = 3x+1$$

$$du = 3dx$$

$$= \frac{1}{3} \int_{0}^{4} u^{3} du$$

$$\int_{1}^{2} \frac{1}{(3-5x)^{2}} dx$$

$$u=3-5x = \int_{3}^{2} u^{-3} \cdot (-\frac{1}{3}dx)$$

$$du=-5dx = -\frac{1}{5} \int_{-3}^{-7} u^{-3} dx$$

$$= -\frac{1}{5} \int_{-3}^{-7} u^{-3} dx$$

$$=$$

Evaluate:

$$\int_{1}^{e} \frac{\ln x}{x} dx = \int_{0}^{1} u du = \frac{u^{3}}{3} \int_{0}^{1} u du = \frac$$

Homework - Exercise 11.3 - pp. 511-512 - Q. 4 and 6 RED BOOK

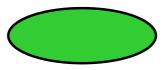
Warm Up

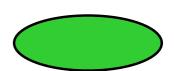
Find:
$$\int 5x^2 \sin(4x^3 + 1) dx$$

$$\int x\sqrt{2x^2-5}\,dx$$

$$\int \cot x dx$$







Differential and Integral Calculus 120

Integration by Parts

As we have discussed before, every differentiation rule has a corresponding integration rule.

The rule that corresponds to the Product Rule for differentiation is called the rule for <u>integration by parts</u>.

The product rule stated that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes... [f(x)g'(x)dx + g(x)f'(x)dx] = f(x)g(x)

or $\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$

which can be rearranged as:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

this formulas above is called

the <u>formula for integration by parts</u>

It is perhaps easier to remember in the following

notation..... Let
$$u = f(x)$$
 and $v = g(x)$ then the differentials are: $du = f'(x)dx$ $dv = g'(x)dx$

And by the Substitution Rule, the formulas becomes...

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Integration By Parts
$$\int u dv = uv - \int v du$$

Let's do an example.... Find: $\int x \sin x dx$

It helps when you stick to this pattern:

we need to make an appropriate choice for u and dv

$$u = \underline{\qquad} dv = \underline{\qquad}$$
$$du = \underline{\qquad} v = \underline{\qquad}$$

Again, the goal in using integration by parts is to obtain a simpler integral than the one we started with... so we must decide on what u and dv are very carefully!

* In general, when deciding on a choice for u and dv, we usually try to choose u = f(x) to be a function that becomes simpler when differentiated...

(or at least \underline{NOT} more complicated) as long as dv = g'(x)dx can be readily integrated to give v.

 $=-x\cos x+\sin x+C$

Find:
$$\int xe^x dx$$

$$= xe^x - e^x + C$$

It helps when you stick to this pattern:

$$u =$$
_____ $dv =$ _____
 $du =$ ____ $v =$ _____

Find:
$$\int x \cos(3x) dx$$

$$=\frac{1}{3}x\sin 3x + \frac{1}{9}\cos 3x + C$$

It helps when you stick to this pattern:

$$u = \underline{\qquad} dv = \underline{\qquad}$$
 $du = \underline{\qquad} v = \underline{\qquad}$