Differential and Integral Calculus 120

$\int$ Substitution Rule! $\int$

Substitution Rule

It often arises that we need to integrate a function which does not follow one of our basic integral formulas therefore other techniques are necessary.

One such technique is called Substitution that involves a change of variable, which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Substitution Rule for Indefinite Integrals

If $u = g(x)$, then $\int f(g(x))g'(x)dx = \int f(u)du$

Substitution Rule for Integration corresponds... to the Chain Rule for Differentiation.

Let's do an example...

Find $\int (x^2 - 5)^8 2x dx$

In using the substitution rule, the idea is to replace a complicated integral by a simpler integral by changing to a new variable u.

In thinking of the appropriate substitution, we try to choose u to be some function in the integrand whose differential du also occurs. (ignoring any constants)

So using the above example we have...

$$\int (x^2 - 5)^8 2x dx$$

Let:

$u = x^2 - 5$

$du = 2x dx$

$$\int (u)^8 du$$

$$= \frac{u^9}{9} + C$$

$$= \frac{(x^2-5)^9}{9} + C$$

Evaluate each of the following:

$$\int \frac{x^2}{\sqrt{1-x^3}}dx$$

$u = 1-x^3$

$du = -3x^2 dx$

$-\frac{1}{3}du = x^2 dx$

$= \int \frac{x^2}{(1-x^3)^{1/2}} dx$

$= \int (u)^{-1/2} \cdot \left(-\frac{1}{3} du\right)$

$= -\frac{1}{3} \int (u)^{-1/2} du$

$= -\frac{1}{3} \cdot 2u^{1/2} + C$

$= -\frac{2}{3}(1-x^3)^{1/2} + C$

$$= -\frac{2}{3}(1-x^3)^{1/2} + C$$

$$\int \sin 4x dx$$

$u = 4x$

$du = 4dx$

$\frac{1}{4}du = dx$

$= \int \sin u \cdot \left(\frac{1}{4} du\right)$

$= \frac{1}{4} \int \sin(u) du$

$= \frac{1}{4} \cdot -\cos u + C$

$= \frac{-1}{4} \cos u + C$

$= -\frac{1}{4} \cos 4x + C$

$$= -\frac{1}{4}\cos 4x + C$$

Evaluate each of the following:

$$\int \frac{\ln x}{x}\,dx = \int \ln x \cdot \frac{1}{x}\,dx$$

$u = \ln x$

$du = \frac{1}{x}\,dx$

$$= \int u\,du$$

$$= \frac{u^2}{2} + C$$

$$= \frac{(\ln x)^2}{2} + C$$

$$= \frac{1}{2}(\ln x)^2 + C$$

$$\int (2+\sin x)^{10} \cos x\,dx$$

$u = 2+\sin x$

$du = \cos x\,dx$

$$= \int u^{10}\,du$$

$$= \frac{u^{11}}{11} + C$$

$$= \frac{(2+\sin x)^{11}}{11} + C$$

$$= \frac{1}{11}(2+\sin x)^{11} + C$$

Evaluate each of the following:

$$\int 2x\sqrt{1+x^2}\,dx$$

$u = 1+x^2$

$du = 2x\,dx$

$= \int u^{1/2}\,du$

$= \frac{2}{3}u^{3/2} + C$

$= \frac{2}{3}(1+x^2)^{3/2} + C$

$$= \frac{2}{3}(x^2+1)^{3/2} + C$$

$$\int x^3\cos(x^4+2)\,dx$$

$u = x^4+2$

$du = 4x^3\,dx$

$\frac{1}{4}du = x^3\,dx$

$= \int \cos(u)\cdot\frac{1}{4}du$

$= \frac{1}{4}\int \cos(u)\,du$

$= \frac{1}{4}\sin u + C$

$= \frac{1}{4}\sin(x^4+2) + C$

$$= \frac{1}{4}\sin(x^4+2) + C$$

Evaluate each of the following:

$$\int \sqrt{2x+1}\,dx$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

$$= \int u^{1/2} \cdot \frac{1}{2}du$$

$$= \frac{1}{2}\int u^{1/2}\,du$$

$$= \frac{1}{2} \cdot \frac{2u^{3/2}}{3} + C$$

$$= \frac{1}{3}u^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$$= \frac{1}{3}(2x+1)^{3/2} + C$$

$$\int \frac{x}{\sqrt{1-4x^2}}\,dx$$

$u = 1-4x^2$

$du = -8x\,dx$

$-\frac{1}{8}du = x\,dx$

$$= \int u^{-1/2} \cdot \left(-\frac{1}{8}du\right)$$

$$= -\frac{1}{8}\int u^{-1/2}\,du$$

$$= -\frac{1}{8} \cdot 2u^{1/2} + C$$

$$= -\frac{1}{4}u^{1/2} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

$$= -\frac{1}{4}\sqrt{1-4x^2} + C$$

Evaluate the following:

$$\int e^{5x}\,dx$$

$u=5x$

$du=5dx$

$\frac{1}{5}du=dx$

$$= \int e^{u}\cdot\left(\frac{1}{5}du\right)$$

$$= \frac{1}{5}\int e^{u}\,du$$

$$= \frac{1}{5}e^{u}+C$$

$$= \frac{1}{5}e^{5x}+C$$

$$= \frac{1}{5}e^{5x}+C$$

Evaluate: $$\int \sqrt{1+x^2}\,x^5\,dx$$

appropriate substitution becomes more obvious if we factor x^5 as $x^4\,x$

$u = 1+x^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

$x^2 = u-1$

$x^4 = (u-1)^2$

$$\int \sqrt{1+x^2}\cdot x^4 \cdot x\,dx$$

$$= \int u^{1/2}\cdot(u-1)^2\cdot\left(\tfrac{1}{2}du\right)$$

$$= \frac{1}{2}\int u^{1/2}\cdot(u-1)^2\,du$$

$$= \frac{1}{2}\int u^{1/2}\left(u^2-2u+1\right)du$$

$$= \frac{1}{2}\int u^{5/2}-2u^{3/2}+u^{1/2}\,du$$

$$= \frac{1}{2}\left(\frac{2}{7}u^{7/2}-\frac{2\cdot 2}{5}u^{5/2}+\frac{2}{3}u^{3/2}\right)+C$$

$$= \frac{1}{7}u^{7/2}-\frac{2}{5}u^{5/2}+\frac{1}{3}u^{3/2}+C$$

$$= \frac{1}{7}\left(1+x^2\right)^{7/2}-\frac{2}{5}\left(1+x^2\right)^{5/2}+\frac{1}{3}\left(1+x^2\right)^{3/2}+C$$

Homework - Exercise 11.3 - pp. 511-512 - Q. 1 - 3

RED BOOK

Let's do one where the higher degree is not under the radical, but outside... how do we handle this????

$$\int x^2\sqrt{8x+5}\,dx$$

$$=\frac{(8x+5)^{\frac{3}{2}}}{672}\left[24x^2-12x+5\right]+C$$

So, $\int x^2\sqrt{8x+5}\,dx$ well let $u=\sqrt{8x+5}$ therefore

$u^2=8x+5$ and $\frac{u^2-5}{8}=x$

So

$2udu=8dx$

And

$\frac{1}{4}udu=dx$

Now let's start the integraon:

$$=\int x^2\sqrt{8x+5}\,dx$$

$$=\int\left(\frac{u^2-5}{8}\right)^2 u\left(\frac{1}{4}udu\right)$$

$$=\int\left(\frac{u^4-10u^2+25}{64}\right)\left(\frac{u^2}{4}du\right)$$

$$=\int\frac{u^6-10u^4+25u^2}{256}du$$

$$=\frac{1}{256}\int u^6-10u^4+25u^2du$$

Now integrate, I brought out the 1/256 (makes it funner) lol.....

$$=\frac{1}{256}\left(\frac{u^7}{7}-2u^5+\frac{25u^3}{3}\right)_{+C}$$

So, now fill in for u and simplify......

$$=\frac{1}{256}\left(\frac{(8x+5)^{\frac{7}{2}}}{7}-2(8x+5)^{\frac{5}{2}}+\frac{25(8x+5)^{\frac{3}{2}}}{3}\right)+C$$

maybe now factor out a 1/21 as well as (8x+5)^3/2

$$=\frac{(8x+5)^{\frac{3}{2}}}{5376}\left[3(8x+5)^2-42(8x+5)+175\right]+C$$

mulply this out and get................

$$=\frac{(8x+5)^{\frac{3}{2}}}{5376}\left[192x^2-96x+40\right]+C$$

and then you can reduce

$$=\frac{(8x+5)^{\frac{3}{2}}}{672}\left[24x^2-12x+5\right]+C$$

amazing!

Questions From Homework

② d) $\int \frac{x+1}{x^2+2x-6}\,dx$

$u = x^2+2x-6$

$du = 2x+2\,dx$

$\frac{1}{2}du = x+1\,dx$

$= \int \frac{1}{u}\cdot\left(\frac{1}{2}du\right)$

$= \frac{1}{2}\int \frac{1}{u}\,du$

$= \frac{1}{2}\ln|u| + C$

When performing a definite integration using substitution, we have to change the limits of integration so that they are the appropriate values of u.

Substitution Rule for Definite Integrals

If $u = g(x)$, then $\int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$

Example...

$$\int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}}dx = \int_1^3 e^u \cdot 2du$$

$u = \sqrt{x} = x^{1/2}$

$du = \frac{1}{2}x^{-1/2}dx = \frac{1}{2\sqrt{x}}dx$

$2du = \frac{1}{\sqrt{x}}dx$

x	u
9	$\sqrt{9} = 3$
1	$\sqrt{1} = 1$

$$= 2\int_1^3 e^u du$$

$$= 2e^u \Big]_1^3$$

$$= 2e^3 - 2e^1 \qquad = 2(e^3 - e)$$

$$= 2e(e^2 - 1)$$

$$= 2e(e+1)(e-1)$$

Find the area under the curve....

$$A = \int_0^1 \frac{1}{2x+1}dx = \int_1^3 \frac{1}{u} \cdot \left(\frac{1}{2}du\right)$$

$u = 2x+1$

$du = 2dx$

$\frac{1}{2}du = dx$

x	u
1	$2(1)+1 = 3$
0	$2(0)+1 = 1$

$$= \frac{1}{2}\int_1^3 \frac{1}{u}du$$

$$= \frac{1}{2}\ln|u| \Big]_1^3$$

$$= \frac{1}{2}\ln 3 - \frac{1}{2}\ln 1 \quad 0$$

$$= \frac{1}{2}\ln 3 - \frac{1}{2}(0)$$

$$= \boxed{\frac{1}{2}\ln 3}$$

$$= \frac{1}{2}\ln 3$$

Evaluate:

$= \frac{26}{3}$

$$\int_0^4 \sqrt{2x+1}\,dx$$

$u = 2x+1$
$du = 2dx$
$\frac{1}{2}du = dx$

x	u
4	9
0	1

$= \int_1^9 u^{1/2}\cdot\left(\frac{1}{2}du\right)$

$= \frac{1}{2}\int_1^9 u^{1/2}du$

$= \frac{1}{\cancel{2}}\cdot\frac{\cancel{2}}{3}u^{3/2}\Big]_1^9$

$= \frac{1}{3}u^{3/2}\Big]_1^9$

$= \frac{1}{3}(9)^{3/2} - \frac{1}{3}(1)^{3/2}$

$= \frac{27}{3} - \frac{1}{3}$

$= \boxed{\frac{26}{3}}$

$= \frac{1}{14}$

$$\int_1^2 \frac{1}{(3-5x)^2}dx$$

$u = 3-5x$
$du = -5dx$
$-\frac{1}{5}du = dx$

*

x	u
2	-7
1	-2

$= \int_{-2}^{-7} u^{-2}\cdot\left(\frac{-1}{5}du\right)$

$= -\frac{1}{5}\int_{-2}^{-7} u^{-2}du$

$= -\frac{1}{5}\cdot -1u^{-1}\Big]_{-2}^{-7}$

$= \frac{1}{5u}\Big]_{-2}^{-7}$

$= \frac{1}{5(-7)} - \frac{1}{5(-2)}$

$= -\frac{1}{35} + \frac{1}{10}$

$= \frac{-2+7}{70}$

$= \frac{5}{70}$

$\boxed{= \frac{1}{14}}$

Evaluate:

$$\int_1^e \frac{\ln x}{x}dx = \int_0^1 u\,du = \frac{u^2}{2}\Big]_0^1$$

$u = \ln x$

$du = \frac{1}{x}dx$

x	u
e	1
1	0

$$= \frac{(1)^2}{2} - \frac{(0)^2}{2}$$

$$= \frac{1}{2} - 0$$

$$\boxed{= \frac{1}{2}}$$

$$= \frac{1}{2}$$

Homework - Exercise 11.3 - pp. 511-512 - Q. 4 and 6

RED BOOK

Warm Up

Find: $\int 5x^2 \sin(4x^3 + 1)dx$

$\int x\sqrt{2x^2 - 5}dx$

$\int \cot x dx$

Differential and Integral Calculus 120

$\int$ Integration by Parts $\int$

As we have discussed before, every differentiation rule has a corresponding integration rule.

The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The product rule stated that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes... $\int [f(x)g'(x)dx + g(x)f'(x)dx] = f(x)g(x)$

or $\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$

which can be rearranged as:

$$\boxed{\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx}$$

this formulas above is called the **formula for integration by parts**

It is perhaps easier to remember in the following notation..... Let $u = f(x)$ and $v = g(x)$

then the differentials are: $du = f'(x)dx$ $dv = g'(x)dx$

And by the Substitution Rule, the formulas becomes...

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Integration By Parts

$$\int u dv = uv - \int v du$$

Let's do an example.... Find: $\int x \sin x dx$

It helps when you stick to this pattern:

$u =$ ______ $dv =$ ______

$du =$ ______ $v =$ ______

we need to make an appropriate choice for u and dv

Again, the goal in using integration by parts is to obtain a simpler integral than the one we started with... so we must decide on what u and dv are very carefully!

✱ In general, when deciding on a choice for u and dv, we usually try to choose u = f(x) to be a function that becomes simpler when differentiated...

(or at least NOT more complicated)

as long as dv = g'(x)dx can be readily integrated to give v.

Let's do an example.... Find: $\int \underbrace{x}_{u} \underbrace{\sin x dx}_{dv}$

It helps when you stick to this pattern:

$u = x$ $dv = \sin x dx$

$du = 1dx$ $v = -\cos x$

$= x(-\cos x) - \int -\cos x \, dx$

$= -x\cos x + \int \cos x dx$

$= -x\cos x + \sin x + C$

$$= -x\cos x + \sin x + C$$

Find: $$\int \underbrace{x}_{u}\underbrace{e^{x}dx}_{dv}$$

It helps when you stick to this pattern:

$u = x$ $\quad dv = e^x dx$

$du = 1dx$ $\quad v = e^x$

$$= xe^x - \int e^x dx$$

$$= xe^x - e^x + C$$

$$= e^x(x-1) + C$$

Find: $$\int \underbrace{x}_{u}\underbrace{\cos(3x)dx}_{dv}$$

It helps when you stick to this pattern:

$u = x$ $\quad dv = \cos(3x)dx$

$du = 1dx$ $\quad v = \frac{1}{3}\sin(3x)$

$$= x\left(\frac{1}{3}\sin(3x)\right) - \int \frac{1}{3}\sin(3x)\,dx$$

$$= \frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin(3x)\,dx$$

$$= \frac{1}{3}x\sin(3x) - \frac{1}{3}\cdot -\frac{1}{3}\cos(3x) + C$$

$$= \frac{1}{3}x\sin(3x) + \frac{1}{9}\cos(3x) + C$$

Find: $$\int \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv}$$

It helps when you stick to this pattern:

$u = \ln x$ $\quad \int dv = \int 1\,dx$

$du = \frac{1}{x}dx$ $\quad v = x$

$$= x\ln x - \int \cancel{x}\,\frac{1}{\cancel{x}}\,dx$$

$$= x\ln x - \int 1\,dx$$

$$= x\ln x - x + C$$

$$= x(\ln x - 1) + C$$

Find: $\int \underbrace{x^2}_{u} \underbrace{\sin(3x)dx}_{dv}$

It helps when you stick to this pattern:

$u = x^2$ $\quad \int dv = \int \sin(3x)dx$

$du = 2x\,dx$ $\quad v = -\frac{1}{3}\cos(3x)$

$= -\frac{1}{3}x^2\cos(3x) - \int \frac{-2x}{3}\cos 3x\,dx$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\int \underbrace{x}_{u}\underbrace{\cos(3x)dx}_{dv}$

$u = x$ $\quad \int dv = \int \cos(3x)dx$

$du = 1dx$ $\quad v = \frac{1}{3}\sin(3x)$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x) - \int \frac{1}{3}\sin 3x\,dx\right]$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x) - \frac{1}{3}\int \sin 3x\,dx\right]$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{3}\left[\frac{1}{3}x\sin(3x) + \frac{1}{9}\cos 3x\right]$

$= -\frac{1}{3}x^2\cos(3x) + \frac{2}{9}x\sin(3x) + \frac{2}{27}\cos(3x) + C$

Find: $$\int \underbrace{x^2}_{u}\,\underbrace{e^x\,dx}_{dv}$$

It helps when you stick to this pattern:

$u = x^2 \qquad \int dv = \int e^x\,dx$

$du = 2x\,dx \qquad v = e^x$

$$= x^2e^x - \int 2xe^x\,dx$$

$$= x^2e^x - \int \underbrace{2x}_{u}\,\underbrace{e^x\,dx}_{dv}$$

$u = 2x \qquad \int dv = \int e^x\,dx$

$du = 2\,dx \qquad v = e^x$

$$= x^2e^x - \left[2xe^x - \int e^x(2\,dx)\right]$$

$$= x^2e^x - 2xe^x + 2\int e^x\,dx$$

$$= x^2e^x - 2xe^x + 2e^x + C$$

$$= e^x(x^2 - 2x + 2) + C$$

We've done this one already,
but let's do it again and evaluate:

It helps when you stick to this pattern:

$u =$ $\ln x$ $\quad \int dv = \int dx$

$du = \frac{1}{x}dx \quad v = x$

$$\int_1^e \underbrace{\ln x}_{u}\, \underbrace{dx}_{dv}$$

$$= x\ln x\Big|_1^e - \left[\int_1^e x \frac{1}{x}\,dx\right]$$

$$= x\ln x\Big|_1^e - \int_1^e dx$$

$$= x\ln x\Big|_1^e - x\Big|_1^e$$

ln e = 1 $\quad$ ln 1 = 0

$$= e\ln e - \ln 1 - [e - 1]$$

$$= e - 0 - e + 1$$

$$= 1$$

$$=1$$

Find: $\int e^x \sin x dx$

It helps when you
stick to this pattern:

$u =$ ______ $dv =$ ______

$du =$ ______ $v =$ ______

Find: $\int e^{x} \cos x dx$

It helps when you
stick to this pattern:

$u =$ ______ $dv =$ ______

$du =$ ______ $v =$ ______

Find: $\int \sin^{-1} x dx$

It helps when you stick to this pattern:

$u =$ ______ $dv =$ ______

$du =$ ______ $v =$ ______

may require substitution rule as well...

Find: $$\int_0^{\frac{\pi}{3}} \sin x \ln(\cos x)dx$$

It helps when you stick to this pattern:

$u =$ ______ $dv =$ ______

$du =$ ______ $v =$ ______

<u>Homework</u> - Exercise 11.4 - pp. 515 - Q. 1,2,4