

Differential and Integral Calculus 120

∫ Substitution Rule! ∫

Substitution Rule

It often arises that we need to integrate a function which does not follow one of our basic integral formulas therefore other techniques are necessary.

One such technique is called Substitution that involves a change of variable, which permits us to rewrite an integrand in a form to which we can apply a basic integration rule.

Substitution Rule for Indefinite Integrals

$$\text{If } u = g(x) \text{ , then } \int f(g(x))g'(x)dx = \int f(u)du$$

Substitution Rule for Integration corresponds...

to the Chain Rule for Differentiation.

Let's do an example...

Find $\int (x^2 - 5)^8 2x dx$

In using the substitution rule, the idea is to replace a complicated integral by a simpler integral by changing to a new variable u .

In thinking of the appropriate substitution, we try to choose u to be some function in the integrand whose differential du also occurs. (ignoring any constants)

So using the above example we have...

Let: $\int (x^2 - 5)^8 2x dx$

$$u = x^2 - 5$$

$$du = 2x dx$$

$$\int (u)^8 du$$

$$= \frac{u^9}{9} + C$$

$$= \frac{(x^2 - 5)^9}{9} + C$$

Evaluate each of the following:

$$\int \frac{x^2}{\sqrt{1-x^3}} dx$$

$$\begin{aligned} u &= 1-x^3 \\ du &= -3x^2 dx \\ -\frac{1}{3} du &= x^2 dx \\ &= \int \frac{x^2}{(1-x^3)^{1/2}} dx \\ &= \int (u)^{-1/2} \cdot \left(-\frac{1}{3} du\right) \\ &= -\frac{1}{3} \int (u)^{-1/2} du \\ &= -\frac{1}{3} \cdot 2u^{1/2} + C \\ &= -\frac{2}{3}(1-x^3)^{1/2} + C \\ &= -\frac{2}{3}(1-x^3)^{1/2} + C \end{aligned}$$

$$\int \sin 4x dx$$

$$\begin{aligned} u &= 4x \\ du &= 4 dx \\ \frac{1}{4} du &= dx \\ &= \int \sin u \cdot \left(\frac{1}{4} du\right) \\ &= \frac{1}{4} \int \sin(u) du \\ &= \frac{1}{4} \cdot -\cos u + C \\ &= -\frac{1}{4} \cos u + C \\ &= -\frac{1}{4} \cos 4x + C \\ &= -\frac{1}{4} \cos 4x + C \end{aligned}$$

Evaluate each of the following:

$$\int \frac{\ln x}{x} dx = \int \ln x \cdot \frac{1}{x} dx$$

$$\begin{aligned} u &= \ln x & = \int u du \\ du &= \frac{1}{x} dx & = \frac{u^2}{2} + C \\ & & = \frac{(\ln x)^2}{2} + C \\ & & = \frac{1}{2} (\ln x)^2 + C \end{aligned}$$

$$\int (2 + \sin x)^{10} \cos x dx$$

$$\begin{aligned} u &= 2 + \sin x & = \int u^{10} du \\ du &= \cos x dx & = \frac{u^{11}}{11} + C \\ & & = \frac{(2 + \sin x)^{11}}{11} + C \\ & & = \frac{1}{11} (2 + \sin x)^{11} + C \end{aligned}$$

Evaluate each of the following:

$$\int \underline{2x} \sqrt{\underline{1+x^2}} \underline{dx}$$

$$\underline{u=1+x^2} = \int \underline{u}^{\frac{1}{2}} du$$

$$\underline{du=2x dx} = \frac{2}{3} u^{\frac{3}{2}} + C$$

$$= \frac{2}{3} (\underline{1+x^2})^{\frac{3}{2}} + C$$

$$= \frac{2}{3} (x^2 + 1)^{3/2} + C$$

$$\int \underline{x^3} \cos(\underline{x^4 + 2}) \underline{dx}$$

$$\underline{u=x^4+2} = \int \cos(u) \cdot \frac{1}{4} du$$

$$\underline{du=4x^3 dx} = \frac{1}{4} \int \cos(u) du$$

$$\underline{\frac{1}{4} du = x^3 dx} = \frac{1}{4} \sin u + C$$

$$= \frac{1}{4} \sin(\underline{x^4+2}) + C$$

$$= \frac{1}{4} \sin(x^4 + 2) + C$$

Evaluate each of the following:

$$\int \sqrt{2x+1} dx$$

$$\begin{aligned} \underline{u=2x+1} &= \int u^{1/2} \cdot \frac{1}{2} du \\ \underline{du=2dx} &= \frac{1}{2} \int u^{1/2} du \\ \underline{\frac{1}{2} du = dx} &= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C \\ &= \frac{1}{3} u^{3/2} + C \\ &= \frac{1}{3} (2x+1)^{3/2} + C \\ &= \frac{1}{3} (2x+1)^{3/2} + C \end{aligned}$$

$$\int \frac{x}{\sqrt{1-4x^2}} dx$$

$$\begin{aligned} \underline{u=1-4x^2} &= \int u^{-1/2} \cdot \left(-\frac{1}{8} du\right) \\ \underline{du=-8x dx} &= -\frac{1}{8} \int u^{-1/2} du \\ \underline{-\frac{1}{8} du = x dx} &= -\frac{1}{8} \cdot 2u^{1/2} + C \\ &= -\frac{1}{4} u^{1/2} + C \\ &= -\frac{1}{4} \sqrt{1-4x^2} + C \\ &= -\frac{1}{4} \sqrt{1-4x^2} + C \end{aligned}$$

Evaluate the following:

$$\int e^{5x} dx$$

$$u = 5x$$

$$du = 5dx$$

$$\frac{1}{5} du = dx$$

$$= \int e^u \cdot \left(\frac{1}{5} du\right)$$

$$= \frac{1}{5} \int e^u du$$

$$= \frac{1}{5} e^u + C$$

$$= \frac{1}{5} e^{5x} + C$$

$$= \frac{1}{5} e^{5x} + C$$

Evaluate:

$$\int \sqrt{1+x^2} x^5 dx$$

appropriate substitution becomes more obvious if we factor x^5 as $x^4 x$

$$\begin{aligned} \underline{u} &= \underline{1+x^2} \\ du &= 2x dx \\ \underline{\frac{1}{2} du} &= \underline{x dx} \end{aligned}$$

$$x^2 = u - 1$$

$$x^4 = (u-1)^2$$

$$\begin{aligned} & \int \underline{\sqrt{1+x^2}} \cdot \underline{x^4} \cdot \underline{x dx} \\ &= \int u^{1/2} \cdot (u-1)^2 \cdot \left(\frac{1}{2} du\right) \\ &= \frac{1}{2} \int u^{1/2} \cdot (u-1)^2 du \\ &= \frac{1}{2} \int u^{1/2} (u^2 - 2u + 1) du \\ &= \frac{1}{2} \int u^{5/2} - 2u^{3/2} + u^{1/2} du \\ &= \frac{1}{2} \left(\frac{2}{7} u^{7/2} - \frac{2 \cdot 2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right) + C \\ &= \frac{1}{7} u^{7/2} - \frac{2}{5} u^{5/2} + \frac{1}{3} u^{3/2} + C \\ &= \frac{1}{7} (1+x^2)^{7/2} - \frac{2}{5} (1+x^2)^{5/2} + \frac{1}{3} (1+x^2)^{3/2} + C \end{aligned}$$

Homework - Exercise 11.3 - pp. 511-512 - Q. 1 - 3
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Let's do one where the higher degree is not under the radical, but outside... how do we handle this????

$$\int x^2 \sqrt{8x+5} dx = \frac{(8x+5)^{\frac{3}{2}}}{672} [24x^2 - 12x + 5] + C$$

So, $\int x^2 \sqrt{8x+5} dx$ well let $u = \sqrt{8x+5}$ therefore

$u^2 = 8x+5$ and $\frac{u^2 - 5}{8} = x$
So

$2u du = 8 dx$
And

$\frac{1}{4} u du = dx$

Now let's start the integraon:

$$\begin{aligned} &= \int x^2 \sqrt{8x+5} dx \\ &= \int \left(\frac{u^2 - 5}{8}\right)^2 u \left(\frac{1}{4} u du\right) \\ &= \int \left(\frac{u^4 - 10u^2 + 25}{64}\right) \left(\frac{u^2}{4} du\right) \\ &= \int \frac{u^6 - 10u^4 + 25u^2}{256} du \\ &= \frac{1}{256} \int u^6 - 10u^4 + 25u^2 du \end{aligned}$$

Now integrate, I brought out the 1/256 (makes it funner) lol....

$$= \frac{1}{256} \left(\frac{u^7}{7} - 2u^5 + \frac{25u^3}{3} \right) + C$$

So, now fill in for u and simplify.....

$$= \frac{1}{256} \left(\frac{(8x+5)^{\frac{7}{2}}}{7} - 2(8x+5)^{\frac{5}{2}} + \frac{25(8x+5)^{\frac{3}{2}}}{3} \right) + C$$

maybe now factor out

a 1/21 as well as (8x+5)^{3/2}

$$= \frac{(8x+5)^{\frac{3}{2}}}{5376} [3(8x+5)^2 - 42(8x+5) + 175] + C$$

mulply this out

and get.....

$$= \frac{(8x+5)^{\frac{3}{2}}}{5376} [192x^2 - 96x + 40] + C$$

and then you can

reduce

$$= \frac{(8x+5)^{\frac{3}{2}}}{672} [24x^2 - 12x + 5] + C$$

amazing!

Questions From Homework

$$\textcircled{a} \text{ d) } \int \frac{x+1}{x^2+2x-6} dx$$

$$u = x^2 + 2x - 6$$

$$du = 2x + 2 dx$$

$$\frac{1}{2} du = x+1 dx$$

$$= \int \frac{1}{u} \cdot \left(\frac{1}{2} du\right)$$

$$= \frac{1}{2} \int \frac{1}{u} du$$

$$= \frac{1}{2} \ln|u| + C$$

When performing a definite integration using substitution, we have to change the limits of integration so that they are the appropriate values of u .

Substitution Rule for Definite Integrals

$$\text{If } u = g(x) \text{ , then } \int_a^b f(g(x))g'(x)dx = \int_{g(a)}^{g(b)} f(u)du$$

Example...

$$\begin{aligned} \int_1^9 \frac{e^{\sqrt{x}}}{\sqrt{x}} dx &= \int_1^3 e^u \cdot 2du \\ &= 2 \int_1^3 e^u du \\ &= 2e^u \Big|_1^3 \\ &= 2e^3 - 2e^1 = 2(e^3 - e) \\ &= 2e(e^2 - 1) \\ &= 2e(e+1)(e-1) \end{aligned}$$

$u = \sqrt{x} = x^{1/2}$
 $du = \frac{1}{2}x^{-1/2} dx = \frac{1}{2\sqrt{x}} dx$
 $2du = \frac{1}{\sqrt{x}} dx$

x	u
9	$\sqrt{9} = 3$
1	$\sqrt{1} = 1$

Find the area under the curve....

$$\begin{aligned} A &= \int_0^1 \frac{1}{2x+1} dx = \int_1^3 \frac{1}{u} \cdot \left(\frac{1}{2} du\right) \\ &= \frac{1}{2} \int_1^3 \frac{1}{u} du \\ &= \frac{1}{2} \ln|u| \Big|_1^3 \\ &= \frac{1}{2} \ln 3 - \frac{1}{2} \ln 1 \leftarrow 0 \\ &= \frac{1}{2} \ln 3 - \frac{1}{2}(0) \\ &= \boxed{\frac{1}{2} \ln 3} \\ &= \frac{1}{2} \ln 3 \end{aligned}$$

$u = 2x+1$
 $du = 2dx$
 $\frac{1}{2} du = dx$

x	u
1	$2(1)+1=3$
0	$2(0)+1=1$

Evaluate:

$$= \frac{26}{3}$$

$$\int_0^4 \sqrt{2x+1} dx$$

$$u = 2x+1$$

$$du = 2dx$$

$$\frac{1}{2} du = dx$$

x	u
4	9
0	1

$$= \int_1^9 u^{1/2} \cdot \left(\frac{1}{2} du\right)$$

$$= \frac{1}{2} \int_1^9 u^{1/2} du$$

$$= \frac{1}{2} \cdot \frac{2}{3} u^{3/2} \Big|_1^9$$

$$= \frac{1}{3} u^{3/2} \Big|_1^9$$

$$= \frac{1}{3} (9)^{3/2} - \frac{1}{3} (1)^{3/2}$$

$$= \frac{27}{3} - \frac{1}{3}$$

$$= \boxed{\frac{26}{3}}$$

$$= \frac{1}{14}$$

$$\int_1^2 \frac{1}{(3-5x)^2} dx$$

$$u = 3-5x$$

$$du = -5dx$$

$$-\frac{1}{5} du = dx$$

x	u
2	-7
1	-2

$$= \int_{-2}^{-7} u^{-2} \cdot \left(-\frac{1}{5} du\right)$$

$$= -\frac{1}{5} \int_{-2}^{-7} u^{-2} du$$

$$= -\frac{1}{5} \cdot \left[-u^{-1} \right]_{-2}^{-7}$$

$$= \frac{1}{5} \Big|_{-2}^{-7}$$

$$= \frac{1}{5(-7)} - \frac{1}{5(-2)}$$

$$= -\frac{1}{35} + \frac{1}{10}$$

$$= \frac{-2+7}{70}$$

$$= \frac{5}{70}$$

$$= \boxed{\frac{1}{14}}$$

Evaluate:

$$\int_1^e \frac{\ln x}{x} dx = \int_0^1 u du = \left. \frac{u^2}{2} \right|_0^1$$

$$u = \ln x$$

$$du = \frac{1}{x} dx$$

$$\begin{array}{r} x \mid u \\ e \mid 1 \\ 1 \mid 0 \end{array}$$

$$= \frac{(1)^2}{2} - \frac{(0)^2}{2}$$

$$= \frac{1}{2} - 0$$

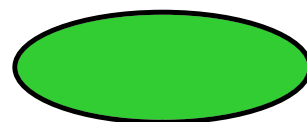
$$= \frac{1}{2}$$

$$= \frac{1}{2}$$

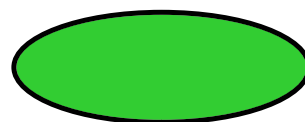
Homework - Exercise 11.3 - pp. 511-512 - Q. 4 and 6
RED BOOK

Warm Up

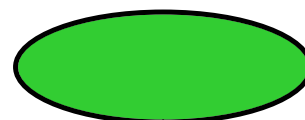
Find: $\int 5x^2 \sin(4x^3 + 1) dx$



$$\int x \sqrt{2x^2 - 5} dx$$



$$\int \cot x dx$$



Differential and Integral Calculus 120

∫ Integration by Parts ∫

As we have discussed before, every differentiation rule has a corresponding integration rule.

The rule that corresponds to the Product Rule for differentiation is called the rule for integration by parts.

The product rule stated that if f and g are differentiable functions, then

$$\frac{d}{dx}[f(x)g(x)] = f(x)g'(x) + g(x)f'(x)$$

In the notation for indefinite integrals this equation becomes... $\int [f(x)g'(x)dx + g(x)f'(x)dx] = f(x)g(x)$

or
$$\int f(x)g'(x)dx + \int g(x)f'(x)dx = f(x)g(x)$$

which can be rearranged as:

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

this formulas above is called

the formula for integration by parts

It is perhaps easier to remember in the following

notation..... Let $u = f(x)$ and $v = g(x)$
 then the differentials are: $du = f'(x)dx$ $dv = g'(x)dx$

And by the Substitution Rule, the formulas becomes...

$$\int f(x)g'(x)dx = f(x)g(x) - \int g(x)f'(x)dx$$

Integration By Parts

$$\int u dv = uv - \int v du$$



Let's do an example.... Find: $\int x \sin x dx$

It helps when you stick to this pattern:

$$u = \underline{\hspace{2cm}} \quad dv = \underline{\hspace{2cm}}$$

$$du = \underline{\hspace{2cm}} \quad v = \underline{\hspace{2cm}}$$

we need to make an appropriate choice for u and dv

Again, the goal in using integration by parts is to obtain a simpler integral than the one we started with... so we must decide on what u and dv are very carefully!

* In general, when deciding on a choice for u and dv , we usually try to choose $u = f(x)$ to be a function that becomes simpler when differentiated... (or at least NOT more complicated) as long as $dv = g'(x)dx$ can be readily integrated to give v .

Let's do an example.... Find: $\int x \sin x dx$

It helps when you stick to this pattern:

$$u = \underline{x} \quad dv = \underline{\sin x dx}$$

$$du = \underline{1 dx} \quad v = \underline{-\cos x}$$

$$= x(-\cos x) - \int -\cos x dx$$

$$= -x \cos x + \int \cos x dx$$

$$= -x \cos x + \underline{\underline{\sin x + C}}$$

$$= -x \cos x + \sin x + C$$

Find: $\int \underbrace{x}_u \underbrace{e^x dx}_v$

It helps when you stick to this pattern:

$$u = \underline{x} \quad dv = \underline{e^x dx}$$

$$du = \underline{1 dx} \quad v = \underline{e^x}$$

$$= x e^x - \underline{\int e^x dx}$$

$$= x e^x - \underline{e^x + C}$$

$$= e^x(x-1) + C$$

Find: $\int \underbrace{x}_u \underbrace{\cos(3x) dx}_v$

It helps when you stick to this pattern:

$$u = \underline{x} \quad dv = \underline{\cos(3x) dx}$$

$$du = \underline{1 dx} \quad v = \underline{\frac{1}{3} \sin(3x)}$$

$$= x \left(\frac{1}{3} \sin(3x) \right) - \underline{\int \frac{1}{3} \sin(3x) dx}$$

$$= \frac{1}{3} x \sin(3x) - \frac{1}{3} \int \underline{\sin(3x) dx}$$

$$= \frac{1}{3} x \sin(3x) - \frac{1}{3} \cdot \frac{1}{3} \cos(3x) + C$$

$$= \frac{1}{3} x \sin(3x) + \frac{1}{9} \cos(3x) + C$$

Find: $\int \ln x dx$

$\underbrace{\ln}_{u} \underbrace{x dx}_{dv}$

It helps when you stick to this pattern:

$$u = \ln x \quad \int dv = \int 1 dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$= x \ln x - \int \cancel{x} \frac{1}{\cancel{x}} dx$$

$$= x \ln x - \int 1 dx$$

$$= x \ln x - \underline{x} + C$$

$$= x(\ln x - 1) + C$$

Find: $\int \underbrace{x^2}_u \underbrace{\sin(3x)}_v dx$

It helps when you stick to this pattern:

$$u = x^2 \quad \int dv = \int \sin(3x) dx$$

$$du = 2x dx \quad v = -\frac{1}{3} \cos(3x)$$

$$= -\frac{1}{3} x^2 \cos(3x) - \int \frac{-2x \cos 3x}{3} dx$$

$$= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \int \underbrace{x \cos(3x)}_u \underbrace{dx}_v$$

$$u = x \quad \int dv = \int \cos(3x) dx$$

$$du = 1 dx \quad v = \frac{1}{3} \sin(3x)$$

$$= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \left[\frac{1}{3} x \sin(3x) - \int \frac{1}{3} \sin 3x dx \right]$$

$$= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \left[\frac{1}{3} x \sin(3x) - \frac{1}{3} \int \sin 3x dx \right]$$

$$= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{3} \left[\frac{1}{3} x \sin(3x) + \frac{1}{9} \cos 3x \right]$$

$$= -\frac{1}{3} x^2 \cos(3x) + \frac{2}{9} x \sin(3x) + \frac{2}{27} \cos(3x) + C$$

Find: $\int x^2 e^x dx$

$\underbrace{x^2}_u \underbrace{e^x}_{dv} dx$

It helps when you stick to this pattern:

$$u = x^2 \quad \int dv = \int e^x dx$$

$$du = 2x dx \quad v = e^x$$

$$u = 2x \quad \int dv = \int e^x dx$$

$$du = 2 dx \quad v = e^x$$

$$= x^2 e^x - \int 2x e^x dx$$

$$= x^2 e^x - \int \underbrace{2x}_u \underbrace{e^x}_{dv} dx$$

$$= x^2 e^x - \left[2x e^x - \int e^x (2 dx) \right]$$

$$= x^2 e^x - 2x e^x + 2 \int e^x dx$$

$$= x^2 e^x - 2x e^x + 2 e^x + C$$

$$= e^x (x^2 - 2x + 2) + C$$

We've done this one already,
but let's do it again and evaluate:

It helps when you
stick to this pattern:

$$u = \frac{\ln x}{x} \quad \int dv = \int dx$$

$$du = \frac{1}{x} dx \quad v = x$$

$$\int_1^e \ln x dx$$

$$= x \ln x \Big|_1^e - \int_1^e x \frac{1}{x} dx$$

$$= x \ln x \Big|_1^e - \int_1^e dx$$

$$= x \ln x \Big|_1^e - x \Big|_1^e$$

$$= e \ln e - \ln 1 - [e - 1]$$

$$= e - 0 - e + 1$$

$$= 1$$

$he=1$ ← (points to $e \ln e$)
 $\ln 1 = 0$ → (points to $\ln 1$)

Find: $\int e^x \sin x dx$

It helps when you
stick to this pattern:

$$u = \underline{\hspace{2cm}} \quad dv = \underline{\hspace{2cm}}$$

$$du = \underline{\hspace{2cm}} \quad v = \underline{\hspace{2cm}}$$

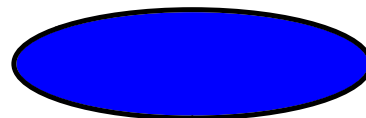


Find: $\int e^x \cos x dx$

It helps when you
stick to this pattern:

$$u = \underline{\hspace{2cm}} \quad dv = \underline{\hspace{2cm}}$$

$$du = \underline{\hspace{2cm}} \quad v = \underline{\hspace{2cm}}$$



Find: $\int \sin^{-1} x dx$

It helps when you
stick to this pattern:

$$u = \underline{\hspace{2cm}} \quad dv = \underline{\hspace{2cm}}$$
$$du = \underline{\hspace{2cm}} \quad v = \underline{\hspace{2cm}}$$

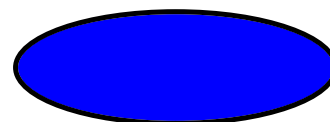
may require substitution rule as well...



Find: $\int_0^{\frac{\pi}{3}} \sin x \ln(\cos x) dx$

It helps when you
stick to this pattern:

$$u = \underline{\hspace{2cm}} \quad dv = \underline{\hspace{2cm}}$$
$$du = \underline{\hspace{2cm}} \quad v = \underline{\hspace{2cm}}$$



Homework - Exercise 11.4 - pp. 515 - Q. 1,2,4