Understanding Logarithms

Chapter 8 (page 370)

$(x,y) \to (y,x)$

reflection in the line $y=x$

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

For the exponential function $y = c^x$, the inverse is $x = c^y$. This inverse is also a function and is called a **logarithmic function**. It is written as $y = \log_c x$, where c is a positive number other than 1. (Page 373)

Logarithmic Form **Exponential Form**

exponent

$\log_c x = y$ $c^y = x$

base

Since our number system is based on powers of 10, **logarithms** with base 10 are widely used and are called **common logarithms**. When you write a common logarithm, you do not need to write the base. For example, log 3 means $\log_{10} 3$.

logarithmic function

- a function of the form $y = \log_c x$, where $c > 0$ and $c \neq 1$, that is the inverse of the exponential function $y = c^x$

logarithm

- an exponent
- in $x = c^y$, y is called the logarithm to base c of x

common logarithm

- a logarithm with base 10

Write each of the following in logarithmic form

a) $32 = 2^5$ exp. answer base

$\boxed{\log_2 (32) = 5}$

$5 = \log_2 32$

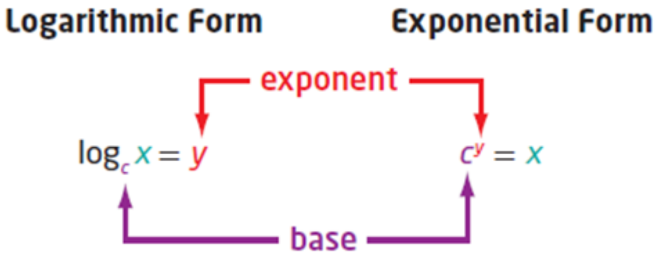
b) $2^{-5} = \dfrac{1}{32}$

$\boxed{\log_2\left(\frac{1}{32}\right) = -5}$

c) $x = 10^y$

$\log_{10} x = y$

$\boxed{\log x = y}$



Write each of the following in exponential form

a) $\log_4 16 = 2$ exp base answer

$\boxed{4^2 = 16}$

$16 = 4^2$

b) $\log_2\left(\dfrac{1}{32}\right) = -5$

$\boxed{2^{-5} = \frac{1}{32}}$

c) $\log 65 = 1.8129$

$\boxed{10^{1.8129} = 65}$

Example 1

Evaluating a Logarithm

Evaluate. (Solving for an exponent)

a) $\log_7 49$ **b)** $\log_6 1$ **c)** $\log 0.001$ **d)** $\log_2 \sqrt{8}$

a) Let $x = \log_7 49$

$x = \log_7 49$

$7^x = 49$ ← express in exponential form → $6^x = 1$

$\cancel{7}^x = \cancel{7}^2$ ← get common base → $\cancel{6}^x = \cancel{6}^0$

$x = 2$

$\therefore \log_7 49 = 2$

b) Let $x = \log_6 1$

$x = \log_6 1$

$6^x = 1$

$\cancel{6}^x = \cancel{6}^0$

$x = 0$

$\therefore \log_6 1 = 0$

c) $x = \log 0.001$

$10^x = 0.001$

$\cancel{10}^x = \cancel{10}^{-3}$

$x = -3$

$\therefore \log 0.001 = -3$

d) $x = \log_2 \sqrt{8}$

$2^x = \sqrt{8}$

$2^x = (8)^{1/2}$

$2^x = (2^3)^{1/2}$

$\cancel{2}^x = \cancel{2}^{3/2}$

$x = \frac{3}{2}$

$\therefore \log_2 \sqrt{8} = \frac{3}{2}$

Example 2

Determine an Unknown in an Expression in Logarithmic Form

Determine the value of x.

a) $\log_5 x = -3$
b) $\log_x 36 = 2$
c) $\log_{64} x = \frac{2}{3}$

a) $\log_5 x = -3$

$5^{-3} = x$

$\left(\frac{1}{5}\right)^3 = x$

$\boxed{\frac{1}{125} = x}$

b) $\log_x 36 = 2$

$x^2 = 36$

$x = \pm 6$

$\boxed{\text{Choose } x = 6}$

c) $\log_{64} x = \frac{2}{3}$

$(64)^{2/3} = x$

$\boxed{16 = x}$

Example 3

Graph the Inverse of an Exponential Function

a) State the inverse of $f(x) = 3^x$.

b) Sketch the graph of the inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

To Find Inverse:

a) $f(x) = 3^x$

$y = 3^x$ (Replace $f(x)$ with y)

$x = 3^y$ (Switch $x + y$)

$\boxed{y = \log_3 x}$ (Solve for y) → Express in logarithmic form

or $f^{-1}(x) = \log_3 x$

$f(x) = 3^x \longrightarrow f^{-1}(x) = \log_3 x$

D: $\{x \mid x \in R\}$	D: $\{x \mid x > 0, x \in R\}$
R: $\{y \mid y > 0, y \in R\}$	R: $\{y \mid y \in R\}$
x-int: none	x-int: $(1,0)$
y-int: $(0,1)$	y-int: none
HA: $y=0$	VA: $x=0$

Solution

a) The inverse of $y = f(x) = 3^x$ is $x = 3^y$ or,

expressed in logarithmic form, $\boxed{y = \log_3 x}$. Since the inverse is a function, it can be written in function notation as $\boxed{f^{-1}(x) = \log_3 x}$.

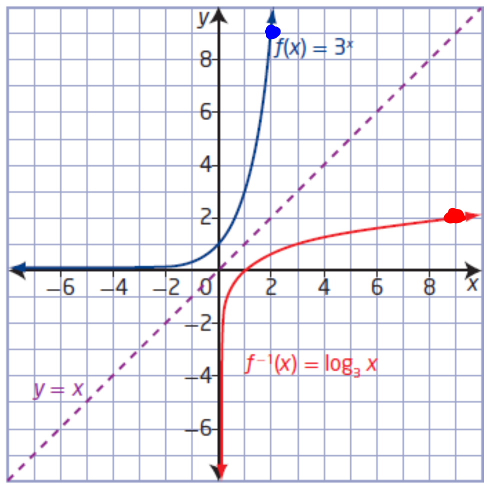
How do you know that $y = \log_3 x$ is a function?

$y=3^x$ passes the horizontal line test

b) Set up tables of values for both the exponential function, $f(x)$, and its inverse, $f^{-1}(x)$. Plot the points and join them with a smooth curve.

$f(x) = 3^x$	
x	y
-3	$\frac{1}{27}$
-2	$\frac{1}{9}$
-1	$\frac{1}{3}$
0	1
1	3
2	9
3	27

$f^{-1}(x) = \log_3 x$	
x	y
$\frac{1}{27}$	-3
$\frac{1}{9}$	-2
$\frac{1}{3}$	-1
1	0
3	1
9	2
27	3



The graph of the inverse, $f^{-1}(x) = \log_3 x$, is a reflection of the graph of $f(x) = 3^x$ about the line $y = x$. For $f^{-1}(x) = \log_3 x$,

- the domain is $\{x \mid x > 0, x \in \mathbb{R}\}$ and the range is $\{y \mid y \in \mathbb{R}\}$
- the x-intercept is 1 (1,0)
- there is no y-intercept
- the vertical asymptote, the y-axis, has equation $x = 0$; there is no horizontal asymptote

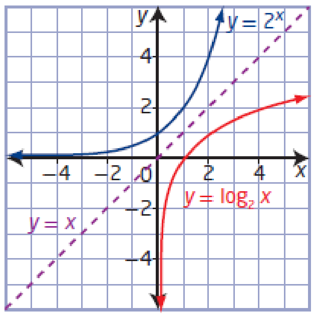
How do the characteristics of $f^{-1}(x) = \log_3 x$ compare to the characteristics of $f(x) = 3^x$?

Key Ideas

- A logarithm is an exponent.
- Equations in exponential form can be written in logarithmic form and vice versa.

Exponential Form	**Logarithmic Form**
$x = c^y$	$y = \log_c x$

- The inverse of the exponential function $y = c^x$, $c > 0$, $c \neq 1$, is $x = c^y$ or, in logarithmic form, $y = \log_c x$. Conversely, the inverse of the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$, is $x = \log_c y$ or, in exponential form, $y = c^x$.
- The graphs of an exponential function and its inverse logarithmic function are reflections of each other in the line $y = x$, as shown.



- For the logarithmic function $y = \log_c x$, $c > 0$, $c \neq 1$,
  - the domain is $\{x \mid x > 0, x \in R\}$
  - the range is $\{y \mid y \in R\}$
  - the x-intercept is 1
  - the vertical asymptote is $x = 0$, or the y-axis
- A common logarithm has base 10. It is not necessary to write the base for common logarithms:

$$\log_{10} x = \log x$$

Questions from Homework

8. a) If $f(x) = 5^x$, state the equation of the inverse, $f^{-1}(x)$.

b) Sketch the graph of $f(x)$ and its inverse. Identify the following characteristics of the inverse graph:
- the domain and range
- the x-intercept, if it exists
- the y-intercept, if it exists
- the equations of any asymptotes

a) Inverse: $f(x)=5^x$

$y=5^x$

$x=5^y$

$y=\log_5 x$

$\boxed{f^{-1}(x)=\log_5 x}$

For the curve $f(x)=5^x$

D: $\{x \mid x \in \mathbb{R}\}$

R: $\{y \mid y>0, y \in \mathbb{R}\}$

x-int: none

y-int: $(0,1)$

HA: $y=0$

b) For the curve $f^{-1}(x)=\log_5 x$

D: $\{x \mid x>0, x \in \mathbb{R}\}$

R: $\{y \mid y \in \mathbb{R}\}$

x-int: $(1,0)$

y-int: none

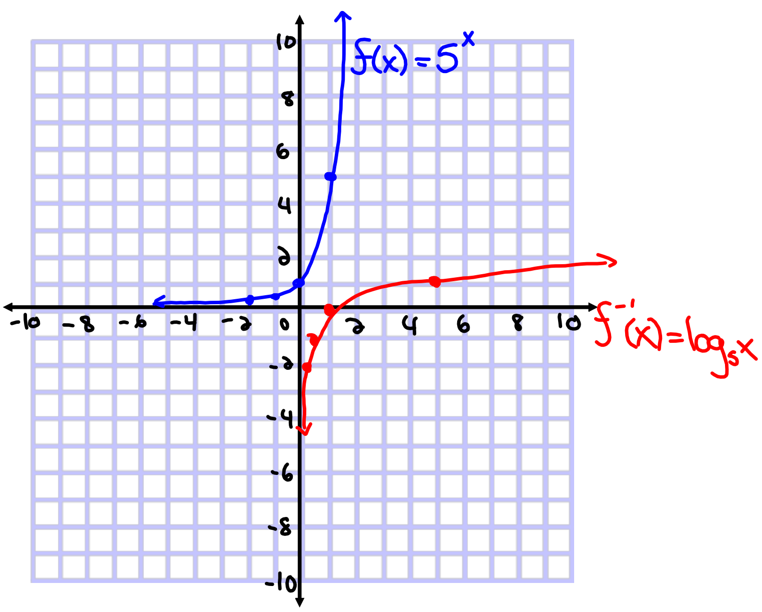
VA: $x=0$

$f(x)=5^x$

x	y
-2	1/25
-1	1/5
0	1
1	5
2	25

$f^{-1}(x)=\log_5 x$

x	y
1/25	-2
1/5	-1
1	0
5	1
25	2



12 b) $\log_x 9 = \frac{1}{2}$ exp

Base ans

$\left(x^{\frac{1}{2}}\right)^{\frac{2}{1}} = (9)^{\frac{2}{1}}$

$x = 81$

d) $\log_x 16 = \frac{4}{3}$

$\left(x^{\frac{4}{3}}\right)^{\frac{3}{4}} = (16)^{\frac{3}{4}}$

$x = 8$

17. The growth of a new social networking site can be modelled by the exponential function $N(t) = 1.1^t$, where N is the number of users after t days.

a) Write the equation of the inverse.

b) How long will it take, to the nearest day, for the number of users to exceed 1 000 000?

a) $N(t) = 1.1^t$

$f(x) = 1.1^x$

$y = 1.1^x$

$x = 1.1^y$

$y = \log_{1.1} x$

$f^{-1}(x) = \log_{1.1} x$

b) $N(t) = 1.1^t$

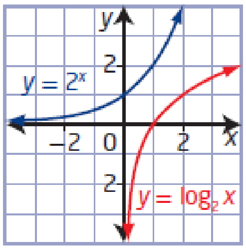
$1000000 = 1.1^t$

$\cancel{1.1}^{144.9} = \cancel{1.1}^t$

$144.9 = t$

8.1 Understanding Logarithms, pages 380 to 382

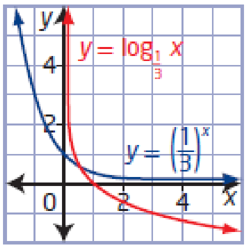
1. a) i)



ii) $y = \log_2 x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

b) i)



ii) $y = \log_{\frac{1}{3}} x$

iii) domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

2. a) $\log_{12} 144 = 2$ **b)** $\log_8 2 = \frac{1}{3}$

c) $\log_{10} 0.000\,01 = -5$ **d)** $\log_7 (y + 3) = 2x$

3. a) $5^2 = 25$ **b)** $8^{\frac{2}{3}} = 4$

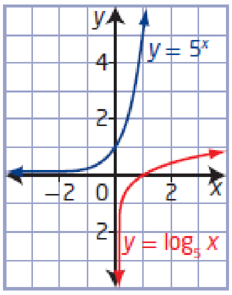
c) $10^6 = 1\,000\,000$ **d)** $11^y = x + 3$

4. a) 3 **b)** 0 **c)** $\frac{1}{3}$ **d)** -3

5. $a = 4$; $b = 5$

8. a) $y = \log_5 x$

b)



domain $\{x \mid x > 0, x \in \mathrm{R}\}$, range $\{y \mid y \in \mathrm{R}\}$, x-intercept 1, no y-intercept, vertical asymptote $x = 0$

10. They are reflections of each other in the line $y = x$.

11. a) They have the exact same shape.

b) One of them is increasing and the other is decreasing.

12. a) 216 **b)** 81 **c)** 64 **d)** 8

13. a) 7 **b)** 6

14. a) 0 **b)** 1

15. -1

16. 16

17. a) $t = \log_{1.1} N$ **b)** 145 days

18. The larger asteroid had a relative risk that was 1479 times as dangerous.

19. 1000 times as great

20. 5

21. $m = 14$, $n = 13$

22. $4n$

23. $y = 3^{2^x}$

Transformations of Logarithmic Functions

Focus on...

- explaining the effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of $y = \log_c x$, where $c > 1$
- sketching the graph of a logarithmic function by applying a set of transformations to the graph of $y = \log_c x$, where $c > 1$, and stating the characteristics of the graph

Remember:

Parameter	Transformation
a	$(x, y) \rightarrow (x, ay)$
b	$(x, y) \rightarrow \left(\frac{x}{b}, y\right)$
h	$(x, y) \rightarrow (x + h, y)$
k	$(x, y) \rightarrow (x, y + k)$

Example 1

Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = \log_3 (x + 9) + 2$.

b) Identify the following characteristics of the graph of the function.

i) the equation of the asymptote **ii)** the domain and range

iii) the y-intercept, if it exists **iv)** the x-intercept, if it exists

a) $a=1$ $b=1$ $h=-9$ $k=2$

$y=3^x$

x	y
-2	1/9
-1	1/3
0	1
1	3
2	9

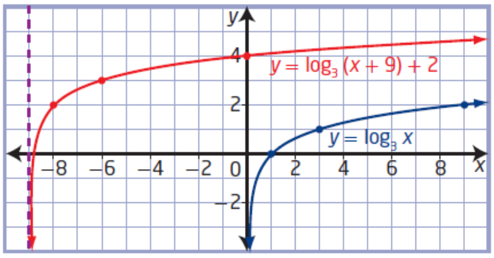
$y=\log_3 x$

x	y
1/9	-2
1/3	-1
1	0
3	1
9	2

$(x,y) \to (x-9, y+2)$

$y=\log_3(x+9)+2$

x	y
-80/9	0
-26/3	1
-8	2
-6	3
0	4



b) (i) VA: $x=-9$ $(h=-9)$

(ii) D: $\{x \mid x>-9, x \in R\}$

R: $\{y \mid y \in R\}$

(iii) y int $(x=0)$

$y=\log_3(x+9)+2$

$y=\log_3(0+9)+2$

$y=\boxed{\log_3 9}+2$

$y=2+2$

$y=4$

$(0,4)$

(iv) x int $(y=0)$

$y=\log_3(x+9)+2$

$0=\log_3(x+9)+2$

$-2=\log_3(x+9)$

$3^{-2}=x+9$

$\frac{1}{9}=x+9$

$\frac{1}{9}-9=x$

$\frac{-80}{9}=x$

$(-80/9, 0)$

Example 2

Reflections, Stretches, and Translations of a Logarithmic Function

a) Use transformations to sketch the graph of the function $y = -\log_2 (2x + 6)$.

b) Identify the following characteristics of the graph of the function.

- **i)** the equation of the asymptote
- **ii)** the domain and range
- **iii)** the y-intercept, if it exists
- **iv)** the x-intercept, if it exists

[redacted]

Key Ideas

- To represent real-life situations, you may need to transform the basic logarithmic function $y = \log_b x$ by applying reflections, stretches, and translations. These transformations should be performed in the same manner as those applied to any other function.
- The effects of the parameters a, b, h, and k in $y = a \log_c (b(x - h)) + k$ on the graph of the logarithmic function $y = \log_c x$ are shown below.

$$y = a \log_c (b(x - h)) + k$$

- a: Vertically stretch by a factor of $|a|$ about the x-axis. Reflect in the x-axis if $a < 0$.
- k: Vertically translate k units.
- b: Horizontally stretch by a factor of $\left|\frac{1}{b}\right|$ about the y-axis. Reflect in the y-axis if $b < 0$.
- h: Horizontally translate h units.

- Only parameter h changes the vertical asymptote and the domain. None of the parameters change the range.

Homework