Understanding Logarithms

$(x,y) \rightarrow (y,x)$
reflection in the line $y=x$

Focus on...

- demonstrating that a logarithmic function is the inverse of an exponential function
- sketching the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- determining the characteristics of the graph of $y = \log_c x$, $c > 0$, $c \neq 1$
- explaining the relationship between logarithms and exponents
- expressing a logarithmic function as an exponential function and vice versa
- evaluating logarithms using a variety of methods

General Properties of Logarithms:

* a logarithm is an exponent !

If $c > 0$ and $c \neq 1$, then...

(i) $\log_c 1 = 0$

(ii) $\log_c c^x = x$

(iii) $c^{\log_c x} = x$

Did You Know?

The input value for a logarithm is called an argument. For example, in the expression $\log_6 1$, the argument is 1.

(i) $\log_5 1 = 0$ (ii) $\log_2 2^3 = 3$ (iii) $7^{\log_7 49} = 49$

$5^{\log_5 10} = 10$

Product Law of Logarithms (Page 394)

The logarithm of a product of numbers can be expressed as the sum of the logarithms of the numbers.

$$\boxed{\log_c MN = \log_c M + \log_c N}$$

Ex: $\log_2 8 + \log_2 3 = \log_2 (8 \times 3) = \boxed{\log_2 24}$

≈ 4.59

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$MN = (c^x)(c^y)$

$MN = c^{x+y}$ Apply the product law of powers.

$\log_c MN = x + y$ Write in logarithmic form.

$\log_c MN = \log_c M + \log_c N$ Substitute for x and y.

Quotient Law of Logarithms

The logarithm of a quotient of numbers can be expressed as the difference of the logarithms of the dividend and the divisor.

$$\boxed{\log_c \frac{M}{N} = \log_c M - \log_c N}$$

Ex: $\log 400 - \log 4 = \log\left(\frac{400}{4}\right) = \log 100$
$= \boxed{2}$

Proof

Let $\log_c M = x$ and $\log_c N = y$, where M, N, and c are positive real numbers with $c \neq 1$.

Write the equations in exponential form as $M = c^x$ and $N = c^y$:

$$\frac{M}{N} = \frac{c^x}{c^y}$$

$$\frac{M}{N} = c^{x-y}$$ Apply the quotient law of powers.

$$\log_c \frac{M}{N} = x - y$$ Write in logarithmic form.

$$\log_c \frac{M}{N} = \log_c M - \log_c N$$ Substitute for x and y.

Power Law of Logarithms

The logarithm of a power of a number can be expressed as the exponent times the logarithm of the number.

$$\boxed{\log_c M^P = P \log_c M}$$

How could you prove the quotient law using the product law and the power law?

Proof

Let $\log_c M = x$, where M and c are positive real numbers with $c \neq 1$.

Write the equation in exponential form as $M = c^x$.

Let P be a real number.

$$M = c^x$$
$$M^P = (c^x)^P$$
$$M^P = c^{xP} \quad \text{Simplify the exponents.}$$
$$\log_c M^P = xP \quad \text{Write in logarithmic form.}$$
$$\log_c M^P = (\log_c M)P \quad \text{Substitute for } x.$$
$$\log_c M^P = P \log_c M$$

The laws of logarithms can be applied to logarithmic functions, expressions, and equations.

Ex: $\log_2 \sqrt{8}$

$\log_2 8^{1/2}$

$\frac{1}{2}\boxed{\log_2 8}$

$\frac{1}{2}(3)$

$\boxed{\frac{3}{2}}$

Questions from Homework

④ c) $\log_{10}(3x+5) = 2$ Logarithmic Form

$10^2 = 3x+5$ Exponential Form

$100 = 3x+5$

$95 = 3x$

$\boxed{\frac{95}{3} = x}$

h) $10^{5^x} = 3$ Exponential Form

$(\log_{10} 3) = 5^x$ ← exp. Logarithmic Form

ans. base

$\boxed{\log_5(\log 3) = x}$

g) $\log_2(\log_3 x) = 4$

$2^4 = \log_3 x$

$16 = \log_3 x$

$3^{16} = x$

$\boxed{43\,046\,721 = x}$

e) $2^{1-x} = 3$

$\log_2 3 = 1-x$

$\boxed{x = 1-\log_2 3}$

Example 1

Use the Laws of Logarithms to Expand Expressions

Write each expression in terms of individual logarithms of x, y, and z.

a) $\log_5 \frac{xy}{z} = \log_5 x + \log_5 y - \log_5 z$

b) $\log_7 \sqrt[3]{x} = \log_7 x^{1/3} = \frac{1}{3}\log_7 x$

c) $\log_6 \frac{1}{x^2} = \log_6 1 - \log_6 x^2 = 0 - 2\log_6 x = -2\log_6 x$

d) $\log \frac{x^3}{y\sqrt{z}}$

$= \log x^3 - [\log y + \log z^{1/2}]$

$= 3\log x - \log y - \frac{1}{2}\log z$

Example 2

Use the Laws of Logarithms to Evaluate Expressions

Use the laws of logarithms to simplify and evaluate each expression.

a) $\log_6 8 + \log_6 9 - \log_6 2$

b) $\log_7 7\sqrt{7}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

a) $\log_6 8 + \log_6 9 - \log_6 2$

$\log_6 \left(\frac{8 \cdot 9}{2}\right)$

$\log_6 36$

$\boxed{2}$

b) $\log_7 7\sqrt{7}$

$\log_7 7 + \log_7 7^{1/2}$

$1 + \frac{1}{2}(1)$

$\frac{2}{2} + \frac{1}{2}$

$\boxed{\frac{3}{2}}$

c) $2 \log_2 12 - \left(\log_2 6 + \frac{1}{3} \log_2 27\right)$

$2 \log_2 12 - \log_2 6 - \frac{1}{3} \log_2 27$

$\log_2 12^2 - \log_2 6 - \log_2 27^{1/3}$

$\log_2 144 - \log_2 6 - \log_2 3$

$\log_2 \left(\frac{144}{6(3)}\right)$

$\log_2 8$

$\boxed{3}$

Example 3

Use the Laws of Logarithms to Simplify Expressions

Write each expression as a single logarithm in simplest form. State the restrictions on the variable.

a) $\log_7 x^2 + \log_7 x - \dfrac{5 \log_7 x}{2}$

b) $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3)$

a) $\log_7 x^2 + \log_7 x - \frac{5}{2}\log_7 x$

$\log_7 x^2 + \log_7 x - \log_7 x^{5/2}$

$\log_7\left(\frac{x^2 \cdot x}{x^{5/2}}\right)$

$\log_7\left(\frac{x^3}{x^{5/2}}\right) \rightarrow 3 - \frac{5}{2}$

$\frac{6}{2} - \frac{5}{2}$

$\frac{1}{2}$

$\log_7 x^{1/2}$

$\boxed{\frac{1}{2}\log_7 x}$, $x>0$

$\log_5(2x-2) - \log_5(x^2+2x-3)$

$\log_5\left(\frac{2x-2}{x^2+2x-3}\right) \leftarrow$ Factor

$\log_5\left[\frac{2(x-1)}{(x^2-x)(3x-3)}\right]$

$\log_5\left[\frac{2(x-1)}{x(x-1)+3(x-1)}\right]$

$\log_5\left(\frac{2\cancel{(x-1)}}{\cancel{(x-1)}(x+3)}\right)$

$\boxed{\log_5\left(\frac{2}{x+3}\right)}$

For the original expression to be defined, both logarithmic terms must be defined.

$2x - 2 > 0$ $\qquad$ $x^2 + 2x - 3 > 0$

$2x > 2$ $\qquad$ $(x + 3)(x - 1) > 0$

$x > 1$ and $x < -3$ or $x > 1$

What other methods could you have used to solve this quadratic inequality?

The conditions $x > 1$ and $x < -3$ or $x > 1$ are both satisfied when $x > 1$.

Hence, the variable x needs to be restricted to $x > 1$ for the original expression to be defined and then written as a single logarithm.

Therefore, $\log_5 (2x - 2) - \log_5 (x^2 + 2x - 3) = \log_5 \frac{2}{x + 3}$, $x > 1$.

Key Ideas

- Let P be any real number, and M, N, and c be positive real numbers with $c \neq 1$. Then, the following laws of logarithms are valid.

Name	Law	Description
Product	$\log_c MN = \log_c M + \log_c N$	The logarithm of a product of numbers is the sum of the logarithms of the numbers.
Quotient	$\log_c \frac{M}{N} = \log_c M - \log_c N$	The logarithm of a quotient of numbers is the difference of the logarithms of the dividend and divisor.
Power	$\log_c M^P = P \log_c M$	The logarithm of a power of a number is the exponent times the logarithm of the number.

- Many quantities in science are measured using a logarithmic scale. Two commonly used logarithmic scales are the decibel scale and the pH scale.

Homework

Finish Exercise 3

Do I really understand??...

a) Express the following as a single logarithm... $2\log_2 3^2 + \log_2 6 - 3\log_2 3$

b) Evaluate the following... $\log_2 (32)^{\frac{1}{3}}$

c) Express the following as a single logarithm... $\frac{1}{2}\left[(\log_5 a + 2\log_5 b) - 3\log_5 c\right]$

d) Express as a single logarithm in simplest form...

$$\frac{3}{4}\left[12(\log_b x^2 - 2\log_b x) + 8\log_b \sqrt{x} - 4\log_b \frac{1}{x^7}\right]$$