

Inverse of a Relation

$f^{-1}(x)$ ← not an exponent

An inverse function is a second function which undoes the work of the first one.

1. Introduction

Suppose we have a function f that takes x to y , so that

$$f(x) = y.$$

An inverse function, which we call f^{-1} , is another function that takes y back to x . So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f , this needs to work for every x that f acts upon.

Inverse of a Relation

The inverse of a relation is found by interchanging the x -coordinates and y -coordinates of the ordered pairs of the relation. In other words, for every ordered pair (x, y) of a relation, there is an ordered pair (y, x) on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line $y = x$.

$(x, y) \rightarrow (y, x)$ In plain English....the x and y coordinates will just switch places

The inverse of a function $y = f(x)$ may be written in the form $x = f(y)$. The inverse of a function is not necessarily a function. When the inverse of f is itself a function, it is denoted as f^{-1} and read as “ f inverse.” When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

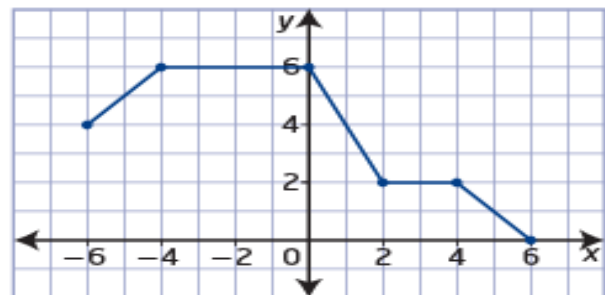
The inverse of a function reverses the processes represented by that function. Functions $f(x)$ and $g(x)$ are inverses of each other if the operations of $f(x)$ reverse all the operations of $g(x)$ in the opposite order and the operations of $g(x)$ reverse all the operations of $f(x)$ in the opposite order.

For example, $f(x) = 2x + 1$ multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is $f^{-1}(x) = \frac{x - 1}{2}$.

Example 1

Graph an Inverse

Consider the graph of the relation shown.

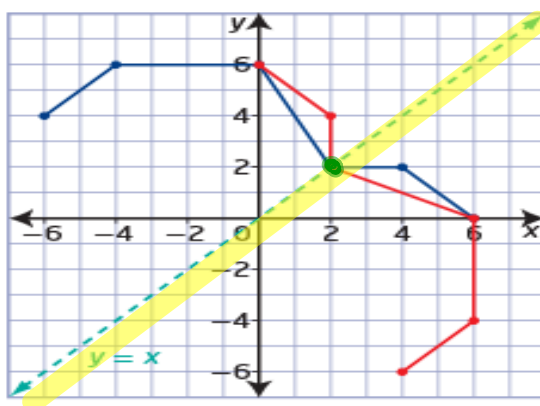


- Sketch the graph of the inverse relation.
- State the domain and range of the relation and its inverse.
- Determine whether the relation and its inverse are functions.

Solution

- To graph the inverse relation, interchange the x -coordinates and y -coordinates of key points on the graph of the relation.

Points on the Relation (x, y)	Points on the Inverse Relation (y, x)
$(-6, 4)$	$(4, -6)$
$(-4, 6)$	$(6, -4)$
$(0, 6)$	$(6, 0)$
$(2, 2)$	$(2, 2)$
$(4, 2)$	$(2, 4)$
$(6, 0)$	$(0, 6)$

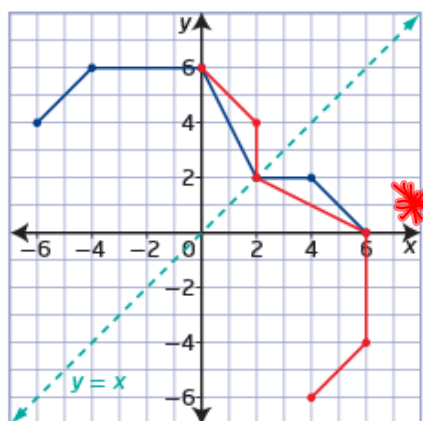


The graphs are reflections of each other in the line $y = x$. The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping $(x, y) \rightarrow (y, x)$.

What points are invariant after a reflection in the line $y = x$?

$(2, 2)$

b) State the domain and range of the relation and its inverse.

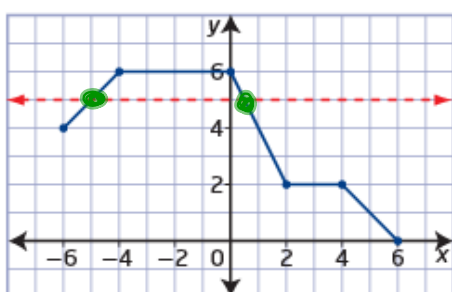


	Domain	Range
Relation	$\{x \mid -6 \leq x \leq 6, x \in \mathbb{R}\}$	$\{y \mid 0 \leq y \leq 6, y \in \mathbb{R}\}$
Inverse Relation	$\{x \mid 0 \leq x \leq 6, x \in \mathbb{R}\}$ or $[0, 6]$	$\{y \mid -6 \leq y \leq 6, y \in \mathbb{R}\}$ or $[-6, 6]$

The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

In plain English....the x and y coordinates will just switch places

c) Determine whether the relation and its inverse are functions.



horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.

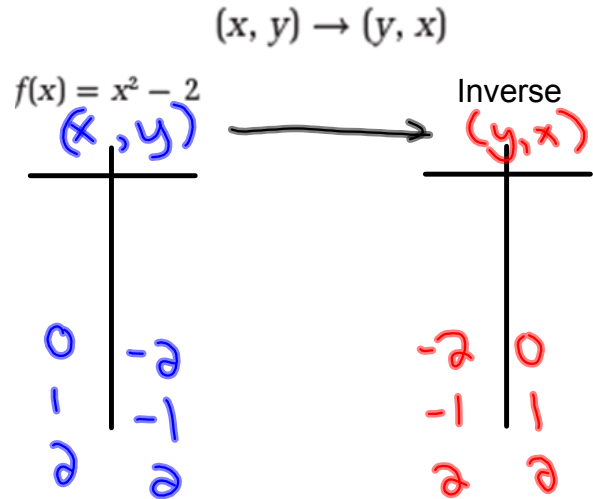
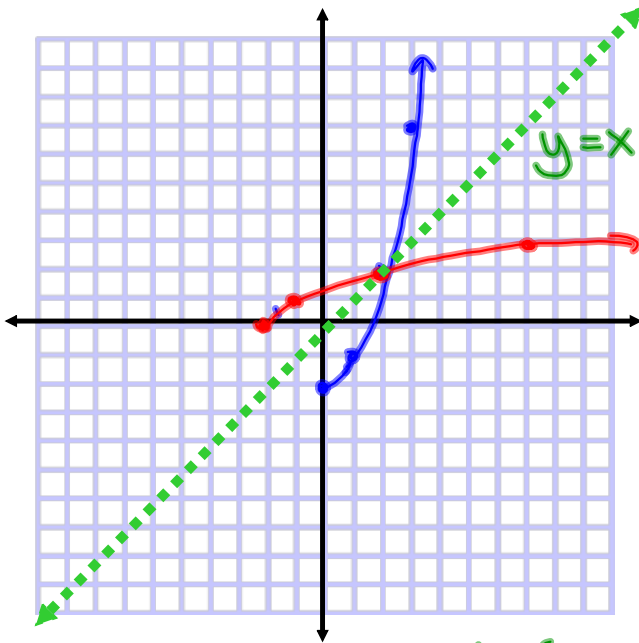
Example 2

Restrict the Domain

$k = -2$ (Down 2)

Consider the function $f(x) = x^2 - 2$.

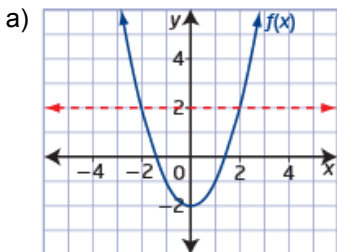
- a) Graph the function $f(x)$. Is the inverse of $f(x)$ a function? **No**
- b) Graph the inverse of $f(x)$ on the same set of coordinate axes.
- c) Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.



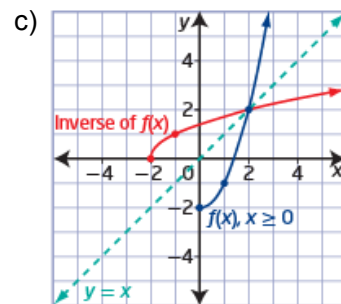
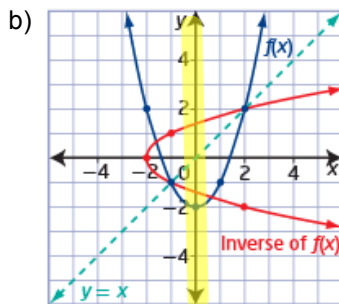
Solutions

only $f(x)$ is a function.

both $f(x)$ and its inverse are functions.



Since $f(x)$ fails the horizontal line test its inverse is not a function



- c) The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test.

One possibility is to restrict the domain of $f(x)$ so that the resulting graph is only one half of the parabola.

Since the equation of the axis of symmetry is $x = 0$, restrict the domain to $\{x \mid x \geq 0, x \in \mathbb{R}\}$.

Example 3

Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function.

Verify graphically that the relations are inverses of each other.

- a) $f(x) = 3x + 6$ → Linear
 b) $f(x) = x^2 - 4$ → Quadratic (Parabola)

$$a) f(x) = 3x + 6$$

$$\textcircled{1} y = 3x + 6$$

$$\textcircled{2} x = 3y + 6$$

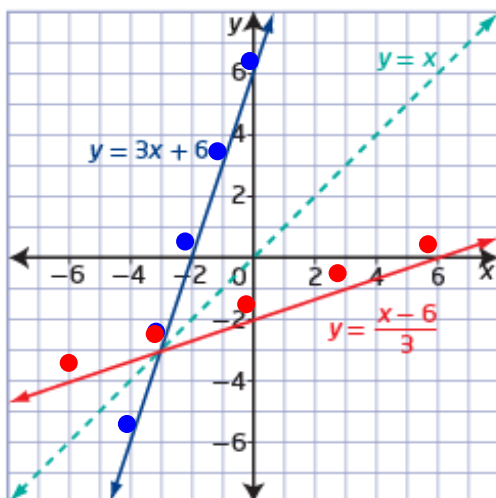
$$\textcircled{3} x - 6 = 3y$$

$$\frac{x-6}{3} = y$$

$$\textcircled{4} f^{-1}(x) = \frac{x-6}{3} = \frac{1}{3}(x-6) = \frac{1}{3}x - 2$$

- 1) Replace $f(x)$ with y .
- 2) Switch x 's and y 's.
- 3) Solve for y .
- 4) Replace y with $f^{-1}(x)$.
(if the inverse is a function!)

Graph $y = 3x + 6$ and $y = \frac{x-6}{3}$ on the same set of coordinate axes.



Determine the Equation of the Inverse

b) $f(x) = x^2 - 4$ → Quadratic (Parabola)

① $y = x^2 - 4$

② $x = y^2 - 4$

③ $x + 4 = y^2$

$\pm \sqrt{x+4} = y$

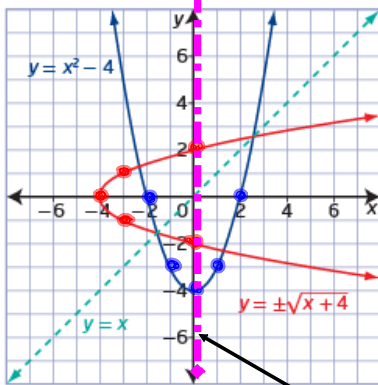
④ $f^{-1}(x) = \sqrt{x+4}$

- | |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|
| <ol style="list-style-type: none"> 1) Replace $f(x)$ with y. 2) Switch x's and y's. 3) Solve for y. 4) Replace y with $f^{-1}(x)$.
<i>(if the inverse is a function!)</i> |
|----------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------------|

Why is this y not replaced with $f^{-1}(x)$? What could be done so that $f^{-1}(x)$ could be used?

Restrict the domain of $f(x)$ so that it will pass the horizontal line test.
 $\{x | x \geq 0, x \in \mathbb{R}\}$ or $\{x | x \leq 0, x \in \mathbb{R}\}$

Graph $y = x^2 - 4$ and $y = \pm\sqrt{x+4}$ on the same set of coordinate axes.



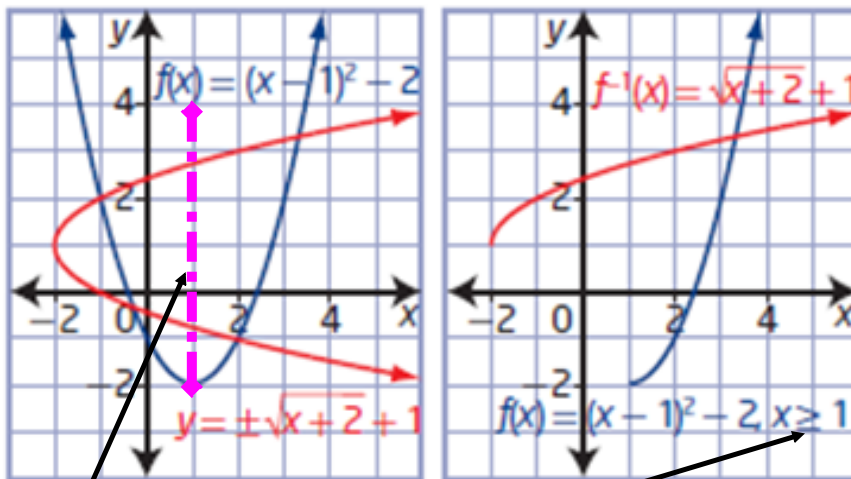
axis of symmetry: $x=0$

$y = x^2 - 4$	→	$y = \pm\sqrt{x+4}$
-2 0		Inverse
-1 -3		0 -2
0 -4		-3 -1
1 -3		-4 0
2 0		-3 1
		0 2

Another example of how to restrict the domain

f) $y = \pm\sqrt{x+2} + 1$

restricted domain
 $\{x \mid x \geq 1, x \in \mathbb{R}\}$



axis of symmetry:
 $x = 1$

Inverse of a Relation

Key Ideas

- You can find the inverse of a relation by interchanging the x -coordinates and y -coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$.
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function $f(x)$ is itself a function, it is denoted by $f^{-1}(x)$.
- You can verify graphically whether two functions are inverses of each other.

Homework

Practice Problems...

Pages 51 - 55

#2, 3, 5, 6, 8, 9, 11, 15, 18, 20, 21

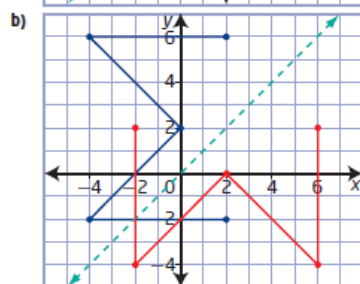
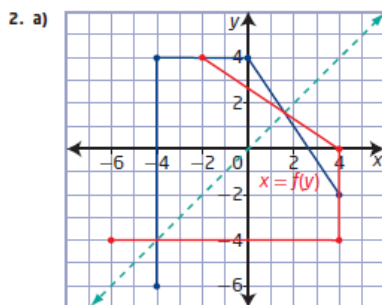
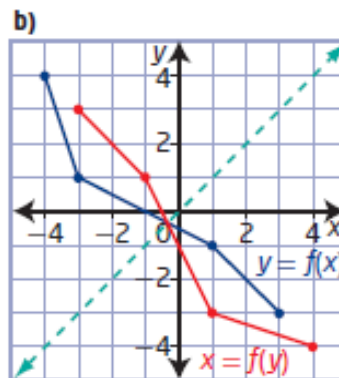
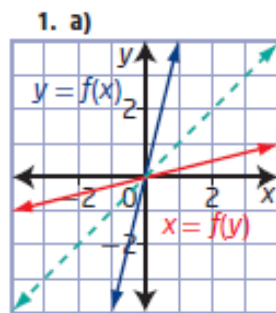
What if given the function algebraically?

Determine algebraically the equation of the inverse of each function.

a) $f(x) = 3x - 6$ **b)** $f(x) = \frac{1}{2}x + 5$

c) $f(x) = \frac{1}{3}(x + 12)$ **d)** $f(x) = \frac{8x + 12}{4}$

1.4 Inverse of a Relation, pages 51 to 55

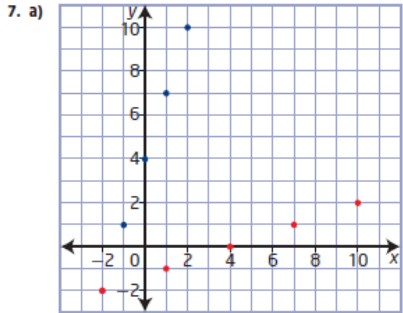


3. a) The graph is a function but the inverse will be a relation.
 b) The graph and its inverse are functions.
 c) The graph and its inverse are relations.

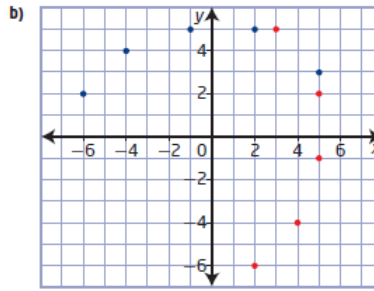
4. Examples:

- a) $\{x \mid x \geq 0, x \in \mathbb{R}\}$ or $\{x \mid x \leq 0, x \in \mathbb{R}\}$
 b) $\{x \mid x \geq -2, x \in \mathbb{R}\}$ or $\{x \mid x \leq -2, x \in \mathbb{R}\}$
 c) $\{x \mid x \geq 4, x \in \mathbb{R}\}$ or $\{x \mid x \leq 4, x \in \mathbb{R}\}$
 d) $\{x \mid x \geq -4, x \in \mathbb{R}\}$ or $\{x \mid x \leq -4, x \in \mathbb{R}\}$
 5. a) $f^{-1}(x) = \frac{1}{7}x$ b) $f^{-1}(x) = -\frac{1}{3}(x - 4)$
 c) $f^{-1}(x) = 3x - 4$ d) $f^{-1}(x) = 3x + 15$
 e) $f^{-1}(x) = -\frac{1}{2}(x - 5)$ f) $f^{-1}(x) = 2x - 6$

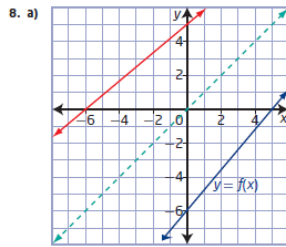
6. a) E b) C c) B d) A e) D



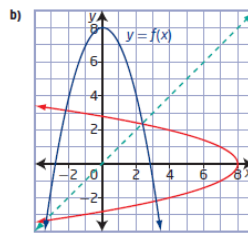
function: domain $\{-2, -1, 0, 1, 2\}$,
range $\{-2, 1, 4, 7, 10\}$
inverse: domain $\{-2, 1, 4, 7, 10\}$,
range $\{-2, -1, 0, 1, 2\}$



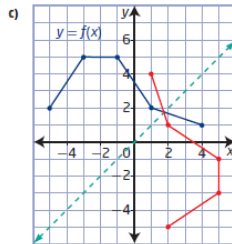
function: domain $\{-6, -4, -1, 2, 5\}$, range $\{2, 3, 4, 5\}$
inverse: domain $\{2, 3, 4, 5\}$, range $\{-6, -4, -1, 2, 5\}$



The inverse is a function; it passes the vertical line test.

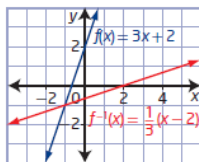


The inverse is not a function; it does not pass the vertical line test.



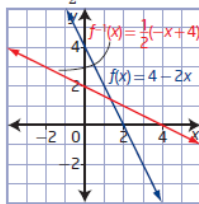
The inverse is not a function; it does not pass the vertical line test.

9. a) $f^{-1}(x) = \frac{1}{3}(x - 2)$



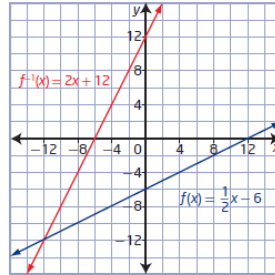
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

b) $f^{-1}(x) = \frac{1}{2}(-x + 4)$



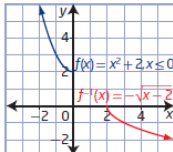
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

c) $f^{-1}(x) = 2x + 12$



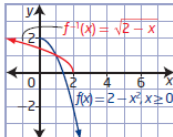
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

d) $f^{-1}(x) = -\sqrt{x-2}$



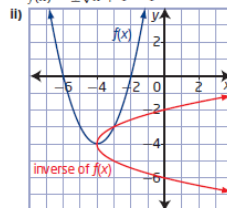
$f(x)$: domain $\{x \mid x \leq 0, x \in \mathbb{R}\}$, range $\{y \mid y \geq 2, y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \geq 2, x \in \mathbb{R}\}$, range $\{y \mid y \leq 0, y \in \mathbb{R}\}$

e) $f^{-1}(x) = \sqrt{2-x}$

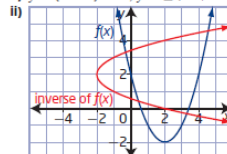


$f(x)$: domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$, range $\{y \mid y \leq 2, y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \leq 2, x \in \mathbb{R}\}$, range $\{y \mid y \geq 0, y \in \mathbb{R}\}$

10. a) i) $f(x) = (x + 4)^2 - 4$, inverse of $f(x) = \pm\sqrt{x+4} - 4$

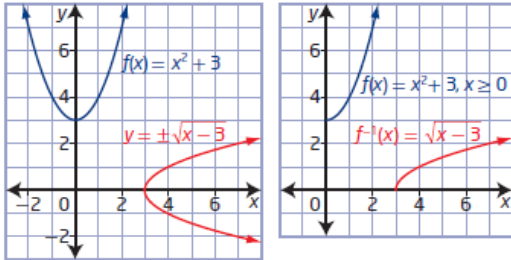


ii) $y = (x - 2)^2 - 2, y = \pm\sqrt{x+2} + 2$

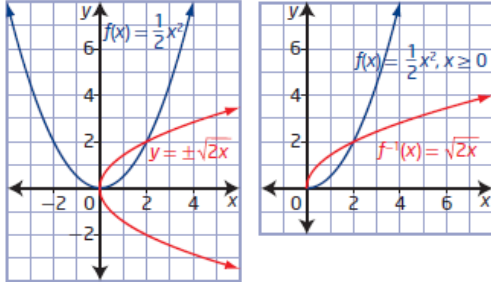


11. Yes, the graphs are reflections of each other in the line $y = x$.

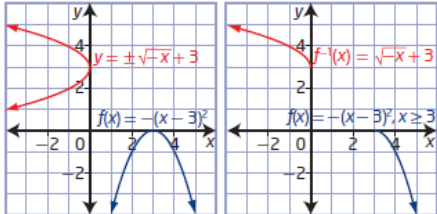
12. a) $y = \pm\sqrt{x-3}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



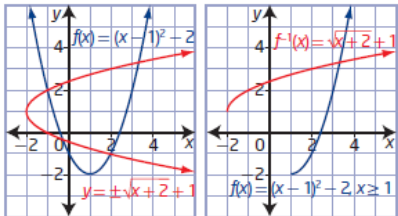
b) $y = \pm\sqrt{2x}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



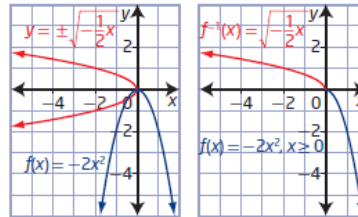
e) $y = \pm\sqrt{-x+3}$ restricted domain $\{x \mid x \geq 3, x \in \mathbb{R}\}$



f) $y = \pm\sqrt{x+2} + 1$ restricted domain $\{x \mid x \geq 1, x \in \mathbb{R}\}$



c) $y = \pm\sqrt{-\frac{1}{2}x}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



d) $y = \pm\sqrt{x-1}$ restricted domain $\{x \mid x \geq -1, x \in \mathbb{R}\}$

