Warm-Up

In how many ways can a teacher seat four girls and three boys in a row of seven seats if a boy must be seated at each end of the row?

3	×	5	×	4	×	3	×	2	×	1	×	2
(Seat 1)		(Seat 2)		(Seat 3)		(Seat 4)		(Seat 5)		(Seat 6)		(Seat 7)

= 720 possible ways to seat the students

Permutations

Focus on...

- ✓ solving counting problems using the fundamental counting principle
- ✓ determining, using a variety of strategies, the number of permutations of *n* elements taken *r* at a time
- solving counting problems when two or more elements are identical
- ✓ solving an equation that involves $_nP_r$ notation

How safe is your password? It has been suggested that a four-character letters-only password can be hacked in under 10 s. However, an eight-character password with at least one number could take up to 7 years to crack. Why is there such a big difference?

The arrangement of objects or people in a line is called a linear **permutation**. In a permutation, the order of the objects is important. When the objects are distinguishable from one another, a new order of objects creates a new permutation.

Seven different objects can be arranged in 7! ways.

$7! = (7)(6)(5)(4)(3)(2)(1)$

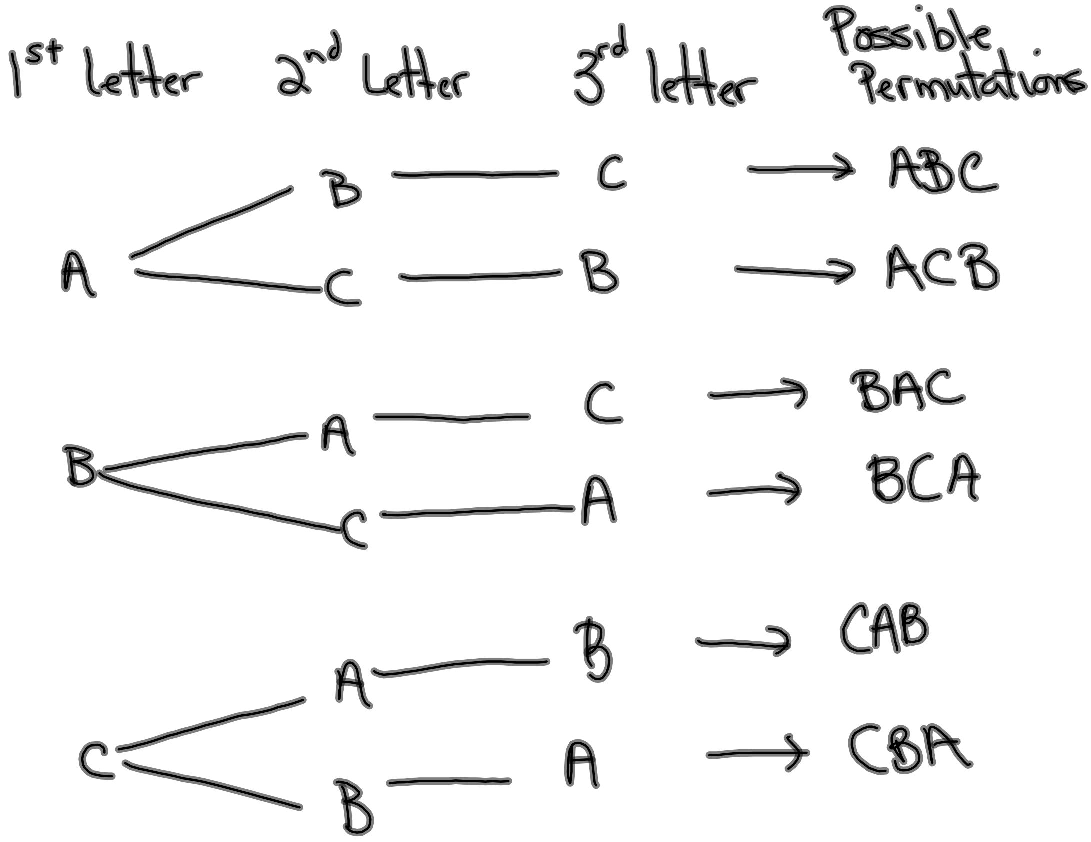
Explain why 7! is equivalent to 7(6!) or to 7(6)(5)(4!).

permutation

- an ordered arrangement or sequence of all or part of a set
- for example, the possible permutations of the letters A, B, and C are ABC, ACB, BAC, BCA, CAB, and CBA

Key Words for permutations:

- in order, in a line, in a row.
- arranged.
- 1st, 2nd, 3rd or Gold, Silver, Bronze



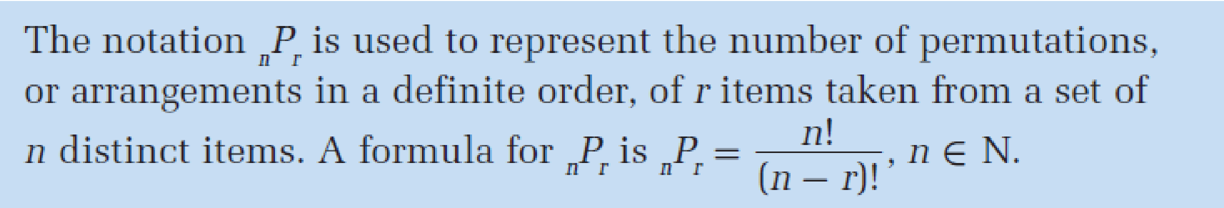
Example

The notation ${}_nP_r$ is used to represent the number of permutations, or arrangements in a definite order, of r items taken from a set of n distinct items. A formula for ${}_nP_r$ is ${}_nP_r = \frac{n!}{(n-r)!}$, $n \in \mathrm{N}$.

If there are seven members on the student council, in how many ways can the council select three students to be the chair, the secretary, and the treasurer of the council?

Using the fundamental counting principle, there are $\underline{(7)(6)(5)}$ possible ways to fill the three positions. Using the factorial notation,

$$\begin{aligned}\frac{7!}{4!} &= \frac{(7)(6)(5)(\overset{1}{\cancel{4}})(\overset{1}{\cancel{3}})(\overset{1}{\cancel{2}})(\overset{1}{\cancel{1}})}{(\underset{1}{\cancel{4}})(\underset{1}{\cancel{3}})(\underset{1}{\cancel{2}})(\underset{1}{\cancel{1}})}\\ &= (7)(6)(5)\\ &= 210\end{aligned}$$



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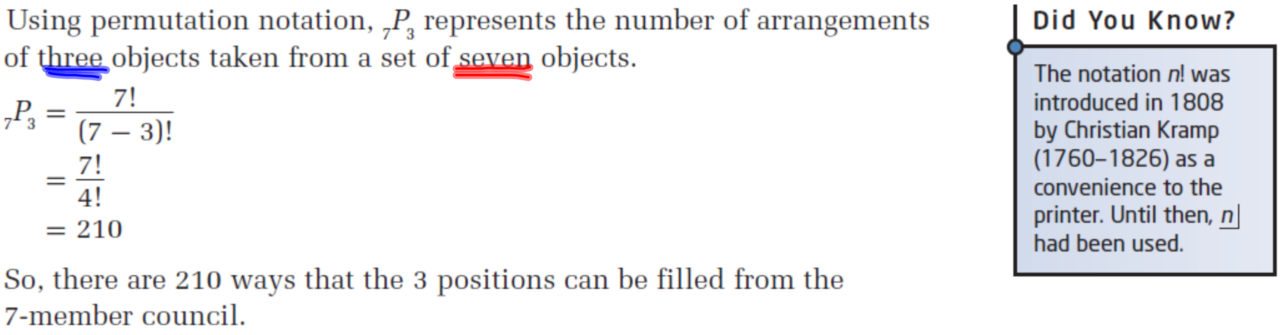
n=7 students   r=3



Using permutation notation, ${}_7P_3$ represents the number of arrangements of three objects taken from a set of seven objects.

$$\begin{aligned} {}_7P_3 &= \frac{7!}{(7-3)!} \\ &= \frac{7!}{4!} \\ &= 210 \end{aligned}$$

So, there are 210 ways that the 3 positions can be filled from the 7-member council.

Did You Know?

The notation $n!$ was introduced in 1808 by Christian Kramp (1760–1826) as a convenience to the printer. Until then, $\underline{n|}$ had been used.

Example 2

Using Factorial Notation

a) Evaluate ${}_9P_4$ using factorial notation.
b) Show that 100! + 99! = 101(99!) without using technology.
c) Solve for n if ${}_nP_3 = 60$, where n is a natural number.

a) $${}_nP_r = \frac{n!}{(n-r)!}$$

$${}_9P_4 = \frac{9!}{(9-4)!} = \frac{9!}{5!} = 9\times8\times7\times6 = 3024$$

b) $$\underline{\underline{100!}} + 99! = 101(99!)$$

group like terms $100(\underline{99!}) + \underline{99!}$

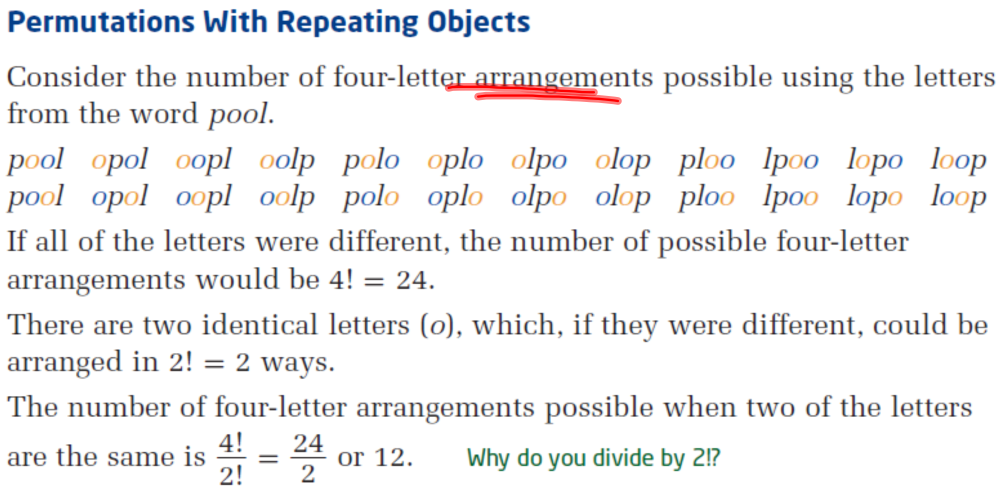
$101(99!)$

In general, a **permutation** is an *arrangement* of objects in different orders, where the **order** of the arrangement **is important**!!!

If "**n**" is the size of the sample space, and "**r**" is the number of items chosen on each trial, then the total number of **permutations** is written as:

$_n\mathbf{P}_r$ and is calculated as $_n\mathbf{P}_r = \dfrac{n!}{(n-r)!}$

Questions from Homework



Permutations With Repeating Objects

Consider the number of four-letter arrangements possible using the letters from the word *pool*.

pool *opol* *oopl* *oolp* *polo* *oplo* *olpo* *olop* *ploo* *lpoo* *lopo* *loop*
pool *opol* *oopl* *oolp* *polo* *oplo* *olpo* *olop* *ploo* *lpoo* *lopo* *loop*

If all of the letters were different, the number of possible four-letter arrangements would be $4! = 24$.

There are two identical letters (*o*), which, if they were different, could be arranged in $2! = 2$ ways.

The number of four-letter arrangements possible when two of the letters are the same is $\frac{4!}{2!} = \frac{24}{2}$ or 12.

Why do you divide by 2!?

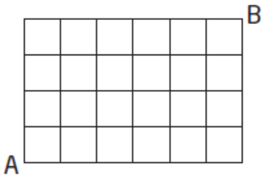
A set of n objects with a of one kind that are identical, b of a second kind that are identical, and c of a third kind that are identical, and so on, can be arranged in $\frac{n!}{a!b!c!\ldots}$ different ways.

Example 3

Repeating Objects

a) How many different eight-letter arrangements can you make using the letters of *aardvark*? $3 \rightarrow$ a's $2 \rightarrow$ r's

b) How many paths can you follow from A to B in a four by six rectangular grid if you move only up or to the right? $4 \rightarrow$ up $6 \rightarrow$ right



Solution

a) There are eight letters in *aardvark*. There are 8! ways to arrange eight letters. But of the eight letters, three are the letter *a* and two are the letter *r*. There are 3! ways to arrange the *a*'s and 2! ways to arrange the *r*'s. The number of different eight-letter arrangements is $\frac{8!}{3!2!} = \underline{\underline{3360}}$.

$$\frac{8!}{3!2!} = \underline{\underline{3360}}$$

b) Each time you travel 1 unit up, it is the same distance no matter where you are on the grid. Similarly, each horizontal movement is the same distance to the right. So, using U to represent 1 unit up and R to represent 1 unit to the right, one possible path is UUUURRRRRR. The problem is to find the number of arrangements of UUUURRRRRR.

The number of different paths is $\frac{10!}{4!6!} = \underline{\underline{210}}$.

Where did the numbers 10, 4, and 6 come from?

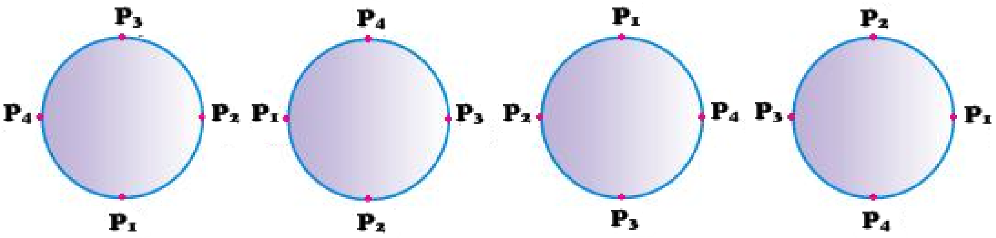
$$\frac{10!}{6!4!} = \underline{\underline{210}}$$

Circular Arrangements

When objects are arranged along a line with first and last place, they form a linear permutation. So far we have dealt only with linear permutations. When objects are arranged along a closed curve or a circle, in which any place may be regarded as the first or last place, they form a **circular permutation**.

The permutation in a row or along a line has a beginning and an end, but there is nothing like beginning or end or first and last in a circular permutation. In circular permutations, we consider one of the objects as fixed and the remaining objects are arranged as in linear permutation.

For example, the following arrangements of 4 people, P_1, P_2, P_3, P_4, in a circle would be considered as the same arrangement.



Here we see that when n = 4, there will be 4 repetitions.

In general, to calculate the number of ways that "n" items can be arranged in a circular fashion, the following formula is used:

$$\frac{{}_nP_n}{n} \quad \textbf{OR} \quad (n-1)!$$

Example

The 12 members of the student council are to be seated at a round table. In how many ways can they be arranged?

Solution

Since n = 12,

$$\begin{array}{lcl}
\dfrac{{}_nP_n}{n} & \qquad \textbf{OR} \qquad & (n-1)! \\
= \dfrac{{}_{12}P_{12}}{12} & & = (12-1)! \\
= \dfrac{479\,001\,600}{12} & & = 11! \\
= 39\,916\,800 & & = 39\,916\,800
\end{array}$$

There are 39 916 800 ways that the members of the student council can sit around the round table.

Example 4

Permutations with Constraints

Five people (A, B, C, D, and E) are seated on a bench. In how many ways can they be arranged if

a) E is seated in the middle? **b)** A and B must be seated together?

c) A and B cannot be together?

* 5 people can be arranged 5! or ${}_5P_5$ or 120 ways

a) $\underline{4} \times \underline{3} \times \underline{1} \times \underline{2} \times \underline{1} = 24$ ways

↑ has to be E

b) • If AB are seated together we can consider them as one item (AB, C, D, E) which can be arranged 4! or 24 ways.

- A + B can be arranged 2! or 2 ways
- $4!2! = (24)(2) = 48$ ways

c) $5! - 4!2! = 120 - 48 = 72$

$5!$ ↑ possible arrangements of 5 objects

$4!2!$ ↑ arrangements where A+B are together

72 ↑ arrangements where A+B are <u>not</u> together

Solution

a) Since E must be in the middle, there is only 1 choice for that position. This leaves four people to be arranged in (4)(3)(2)(1) ways.

4	3	1	2	1
(Seat 1)	(Seat 2)	(Seat 3)	(Seat 4)	(Seat 5)
		Middle		

What is the restriction?

There are (4)(3)(1)(2)(1) = 24 ways to arrange the five people with E seated in the middle.

b) There are 2! ways to arrange A and B together, AB or BA.
Consider A and B together as 1 object. This means that there are 4 objects (C, D, E, and AB) to arrange in 4! = 24 ways.
Then, there are 2!4! = 48 ways to arrange five people if A and B must be seated together.

c) **Method 1: Use Positions When A and B Are Not Together**
There are five positions on the bench. A and B are not together when they are in the following positions:

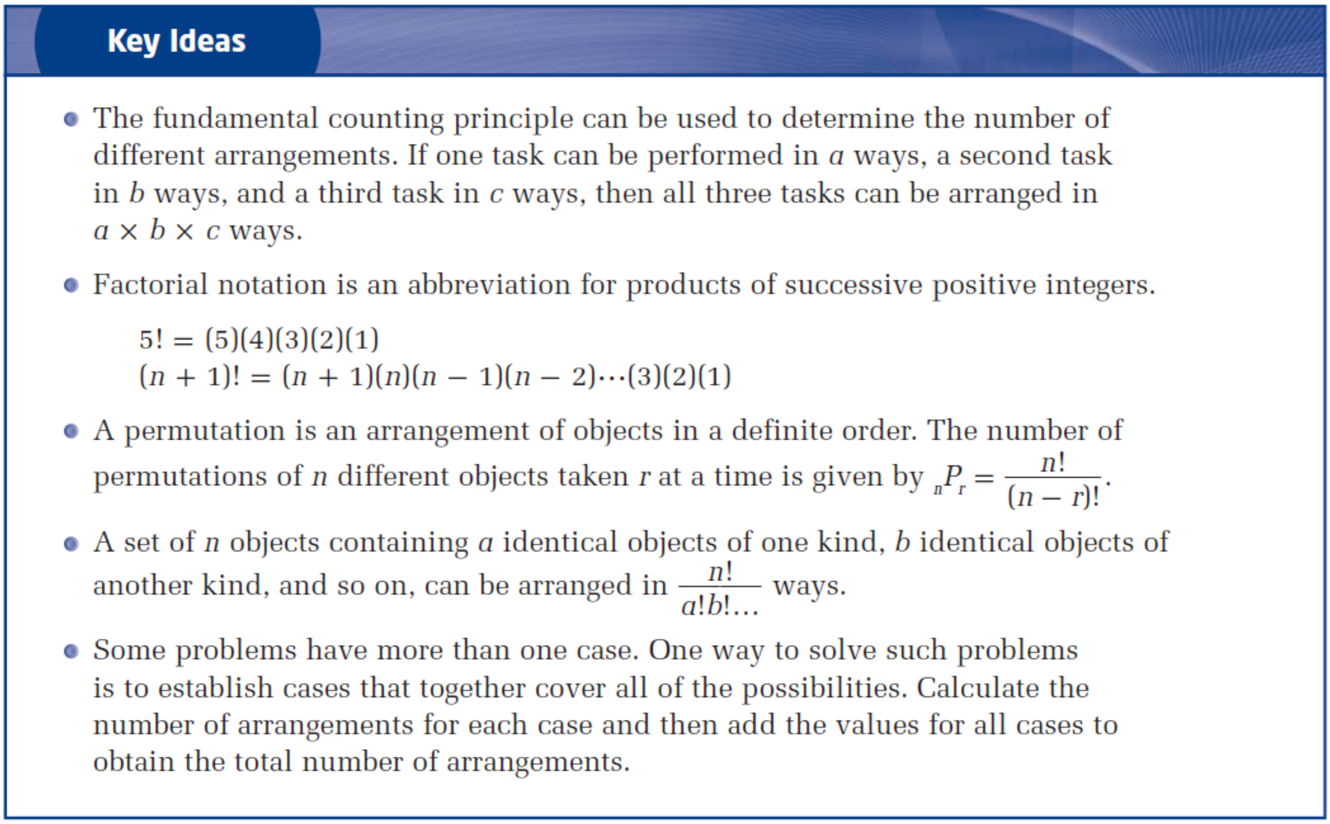
1st and 3rd 1st and 4th 1st and 5th
2nd and 4th 2nd and 5th 3rd and 5th (6 ways)

For any one of these six arrangements, A and B can be interchanged. (2 ways)

The remaining 3 people can always be arranged 3! or 6 ways. (6 ways)
There are (6)(2)(6) = 72 ways where A and B are not seated together.

Why is it necessary to multiply to get the final answer?

Method 2: Use Positions When A and B Are Together
The total number of arrangements for five people in a row with no restrictions is 5! = 120. Arrangements with A and B together is 48 from part b).
Therefore, the number of arrangements with A and B not together is
Total number of arrangements − Number of arrangements together
= 5! − 2!4!
= 120 − 48
= 72



Key Ideas

- The fundamental counting principle can be used to determine the number of different arrangements. If one task can be performed in a ways, a second task in b ways, and a third task in c ways, then all three tasks can be arranged in $a \times b \times c$ ways.
- Factorial notation is an abbreviation for products of successive positive integers.

$$5! = (5)(4)(3)(2)(1)$$
$$(n + 1)! = (n + 1)(n)(n - 1)(n - 2)\cdots(3)(2)(1)$$

- A permutation is an arrangement of objects in a definite order. The number of permutations of n different objects taken r at a time is given by ${}_nP_r = \dfrac{n!}{(n - r)!}$.
- A set of n objects containing a identical objects of one kind, b identical objects of another kind, and so on, can be arranged in $\dfrac{n!}{a!b!\ldots}$ ways.
- Some problems have more than one case. One way to solve such problems is to establish cases that together cover all of the possibilities. Calculate the number of arrangements for each case and then add the values for all cases to obtain the total number of arrangements.

Homework

Answers to Homework