Radical Functions and Transformations

Focus on...

- investigating the function $y = \sqrt{x}$ using a table of values and a graph
- graphing radical functions using transformations
- identifying the domain and range of radical functions

radical function

- a function that involves a radical with a variable in the radicand
- $y = \sqrt{3x}$ and $y = 4\sqrt[3]{5 + x}$ are radical functions.

Example 1

Graph Radical Functions Using Tables of Values

$y = a\sqrt{b(x-h)} + k$

Use a table of values to sketch the graph of each function. Then, state the domain and range of each function.

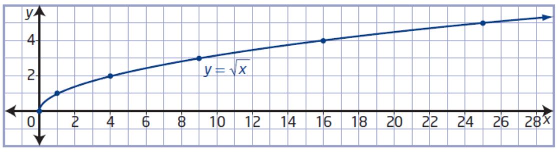
a) $y = \sqrt{x}$ **b)** $y = \sqrt{x - 2}$ **c)** $y = \sqrt{x} - 3$

(base function)

a) For the function $y = \sqrt{x}$, the radicand x must be greater than or equal to zero, $x \geq 0$. (cannot take the square root of a negative)

x	y
0	0
1	1
4	2
9	3
16	4
25	5

How can you choose values of x that allow you to complete the table without using a calculator? → Use Perfect squares



The graph has an endpoint at (0, 0) and continues up and to the right. The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

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b) For the function $y = \sqrt{x - 2}$, the value of the radicand must be greater than or equal to zero.

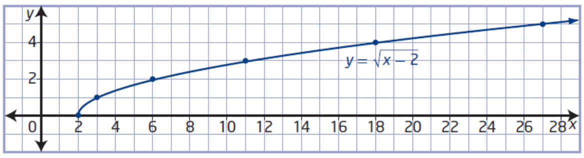
$h = 2$
Translated 2 units to the right

$x - 2 \geq 0$
$x \geq 2$

x	y
2	0
3	1
6	2
11	3
18	4
27	5

How is this table related to the table for $y = \sqrt{x}$ in part a)?

How does the graph of $y = \sqrt{x - 2}$ compare to the graph of $y = \sqrt{x}$?



The domain is $\{x \mid x \geq 2, x \in \mathrm{R}\}$. The range is $\{y \mid y \geq 0, y \in \mathrm{R}\}$.

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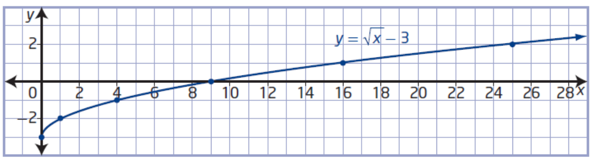
c) The radicand of $y = \sqrt{x} - 3$ must be non-negative.
$x \geq 0$

$k = -3$
Translated 3 units down

x	y
0	−3
1	−2
4	−1
9	0
16	1
25	2

How does the graph of $y = \sqrt{x} - 3$ compare to the graph of $y = \sqrt{x}$?



The domain is $\{x \mid x \geq 0, x \in \mathrm{R}\}$ and the range is $\{y \mid y \geq -3, y \in \mathrm{R}\}$.

Graphing Radical Functions Using Transformations

You can graph a radical function of the form $y = a\sqrt{b(x - h)} + k$ by transforming the graph of $y = \sqrt{x}$ based on the values of a, b, h, and k. The effects of changing parameters in radical functions are the same as the effects of changing parameters in other types of functions.

- Parameter a results in a vertical stretch of the graph of $y = \sqrt{x}$ by a factor of $|a|$. If $a < 0$, the graph of $y = \sqrt{x}$ is reflected in the x-axis.
- Parameter b results in a horizontal stretch of the graph of $y = \sqrt{x}$ by a factor of $\frac{1}{|b|}$. If $b < 0$, the graph of $y = \sqrt{x}$ is reflected in the y-axis.
- Parameter h determines the horizontal translation. If $h > 0$, the graph of $y = \sqrt{x}$ is translated to the right h units. If $h < 0$, the graph is translated to the left $|h|$ units.
- Parameter k determines the vertical translation. If $k > 0$, the graph of $y = \sqrt{x}$ is translated up k units. If $k < 0$, the graph is translated down $|k|$ units.

Example 2

Graph Radical Functions Using Transformations

$y = a\sqrt{b(x-h)} + k$

Sketch the graph of each function using transformations. Compare the domain and range to those of $y = \sqrt{x}$ and identify any changes.

a) $y = 3\sqrt{-(x-1)}$ **b)** $y - 3 = -\sqrt{2x}$

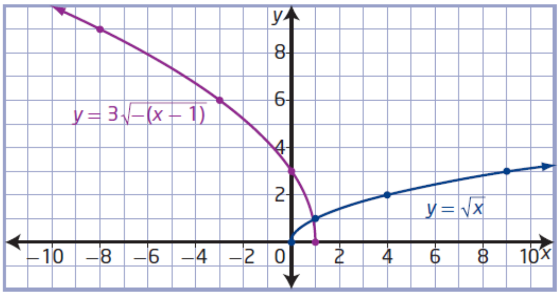
a) $y = 3\sqrt{-(x-1)}$

$a=3$ $b=-1$ $h=1$ $k=0$

x	y
0	0
1	1
4	2
9	3
16	4
25	5

$(x, y) \longrightarrow [-1x+1, 3y+0]$

x	y
1	0
0	3
-3	6
-8	9
-15	12
-24	15



Domain:
$\{x \mid x \leq 1, x \in R\}$

$-(x-1) \geq 0$
$-x+1 \geq 0$
$-x \geq -1$
$x \leq 1$

Range:
$\{y \mid y \geq 0, y \in R\}$

b) $y - 3 = -\sqrt{2x}$

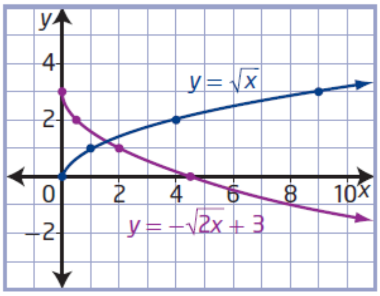
$y = -\sqrt{2x} + 3$

$a = -1$ $b = 2$ $h = 0$ $k = 3$

x	y
0	0
1	1
4	2
9	3
16	4
25	5

$(x, y) \longrightarrow \left[\frac{x}{2} + 0, -1y + 3\right]$

x	y
0	3
0.5	2
2	1
4.5	0
8	-1
12.5	-2



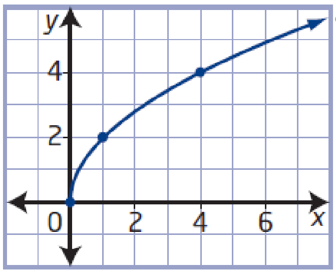
Domain:
$\{x \mid x \geq 0, x \in R\}$

Range:
$\{y \mid y \leq 3, y \in R\}$

Example 3

Determine a Radical Function From a Graph

Mayleen is designing a symmetrical pattern. She sketches the curve shown and wants to determine its equation and the equation of its reflection in each quadrant. The graph is a transformation of the graph of $y = \sqrt{x}$. What are the equations of the four functions Mayleen needs to work with?



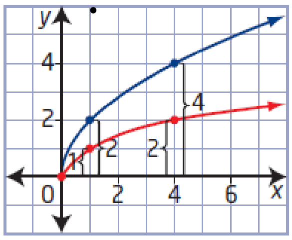
A radical function that involves a stretch can be obtained from either a vertical stretch or a horizontal stretch. Use an equation of the form $y = a\sqrt{x}$ or $y = \sqrt{bx}$ to represent the image function for each type of stretch.

Method 1: Compare Vertical or Horizontal Distances

Superimpose the graph of $y = \sqrt{x}$ and compare corresponding distances to determine the factor by which the function has been stretched.

View as a Vertical Stretch ($y = a\sqrt{x}$)

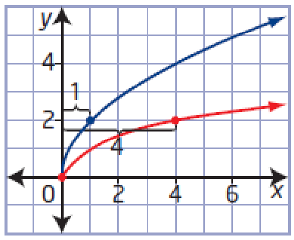
Each vertical distance is 2 times the corresponding distance for $y = \sqrt{x}$.



This represents a vertical stretch by a factor of 2, which means $a = 2$. The equation $y = 2\sqrt{x}$ represents the function.

View as a Horizontal Stretch ($y = \sqrt{bx}$)

Each horizontal distance is $\frac{1}{4}$ the corresponding distance for $y = \sqrt{x}$.



This represents a horizontal stretch by a factor of $\frac{1}{4}$, which means $b = 4$. The equation $y = \sqrt{4x}$ represents the function.

Express the equation of the function as either $y = 2\sqrt{x}$ or $y = \sqrt{4x}$.

$y = 2\sqrt{x}$ → quadrant 1

$y = 2\sqrt{-x}$ → quadrant 2

$y = -2\sqrt{-x}$ → quadrant 3

$y = -2\sqrt{x}$ → quadrant 4

Example 4

Model the Speed of Sound

Justin's physics textbook states that the speed, s, in metres per second, of sound in dry air is related to the air temperature, T, in degrees Celsius, by the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$.

a) Determine the domain and range in this context.
b) On the Internet, Justin finds another formula for the speed of sound, $s = 20\sqrt{T + 273}$. Use algebra to show that the two functions are approximately equivalent.
c) How is the graph of this function related to the graph of the base square root function? Which transformation do you predict will be the most noticeable on a graph?
d) Graph the function $s = 331.3\sqrt{1 + \frac{T}{273.15}}$ using technology.
e) Determine the speed of sound, to the nearest metre per second, at each of the following temperatures.
- **i)** 20 °C (normal room temperature)
- **ii)** 0 °C (freezing point of water)
- **iii)** −63 °C (coldest temperature ever recorded in Canada)
- **iv)** −89 °C (coldest temperature ever recorded on Earth)

a) Domain: $1 + \frac{T}{273.15} \geq 0$ $\quad \{T \mid T \geq -273.15, T \in R\}$

$\frac{T}{273.15} \geq -1$

$T \geq -273.15°$

Range: $\{s \mid s \geq 0, s \in R\}$

b) $s = 331.3\sqrt{\frac{1}{1} + \frac{T}{273.15}}$

$s = 331.3\sqrt{\frac{273.15}{273.15} + \frac{T}{273.15}}$

$s = 331.3\sqrt{\frac{273.15 + T}{273.15}}$

$s = 331.3\frac{\sqrt{273.15 + T}}{\sqrt{273.15}}$

$s = \frac{331.3\sqrt{273.15 + T}}{16.527}$

$s = 20.04\sqrt{273.15 + T}$

$s \approx 20\sqrt{T + 273}$

The graph of $s = \sqrt{T}$ is stretched vertically by a factor of about 20 and then translated about 273 units to the left. Translating 273 units to the left will be most noticeable on the graph of the function.

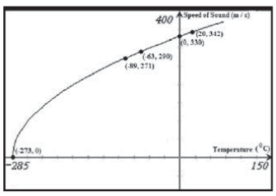
Are these transformations consistent with the domain and range?

c) $s = 20\sqrt{T + 273}$

$a = 20 \rightarrow$ vertical stretch by a factor of 20

$h = -273 \rightarrow$ translation of 273 units to the left

d)



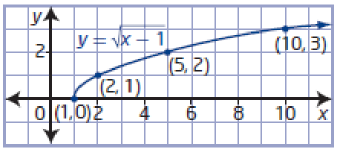
Are your answers to part c) confirmed by the graph?

e)

	Temperature (°C)	Approximate Speed of Sound (m/s)
i)	20	343
ii)	0	331
iii)	−63	291
iv)	−89	272

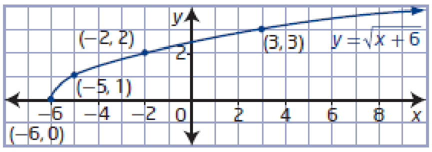
2.1 Radical Functions and Transformations, pages 72 to 77

1. a)



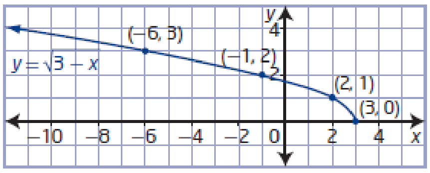
domain $\{x \mid x \geq 1, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

b)



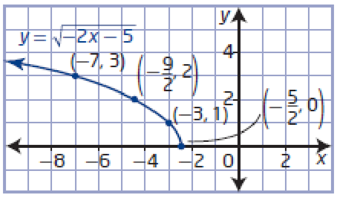
domain $\{x \mid x \geq -6, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

c)



domain $\{x \mid x \leq 3, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

d)



domain $\left\{x \mid x \leq -\frac{5}{2}, x \in \mathrm{R}\right\}$,
range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

2. a) $a = 7 \rightarrow$ vertical stretch by a factor of 7
$h = 9 \rightarrow$ horizontal translation 9 units right
domain $\{x \mid x \geq 9, x \in \mathrm{R}\}$, range $\{y \mid y \geq 0, y \in \mathrm{R}\}$

b) $b = -1 \rightarrow$ reflected in y-axis
$k = 8 \rightarrow$ vertical translation up 8 units
domain $\{x \mid x \leq 0, x \in \mathrm{R}\}$, range $\{y \mid y \geq 8, y \in \mathrm{R}\}$

c) $a = -1 \rightarrow$ reflected in x-axis
$b = \frac{1}{5} \rightarrow$ horizontal stretch factor of 5
domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$, range $\{y \mid y \leq 0, y \in \mathrm{R}\}$

d) $a = \frac{1}{3} \rightarrow$ vertical stretch factor of $\frac{1}{3}$
$h = -6 \rightarrow$ horizontal translation 6 units left
$k = -4 \rightarrow$ vertical translation 4 units down
domain $\{x \mid x \geq -6, x \in \mathrm{R}\}$,
range $\{y \mid y \geq -4, y \in \mathrm{R}\}$

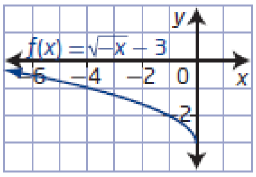
3. a) B **b)** A **c)** D **d)** C

4. a) $y = 4\sqrt{x + 6}$ **b)** $y = \sqrt{8x} - 5$

c) $y = \sqrt{-(x - 4)} + 11$ or $y = \sqrt{-x + 4} + 11$

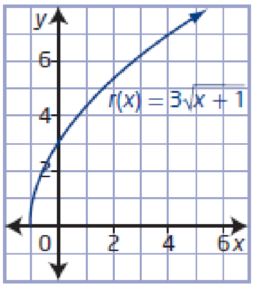
d) $y = -0.25\sqrt{0.1x}$ or $y = -\frac{1}{4}\sqrt{\frac{1}{10}x}$

5. a)



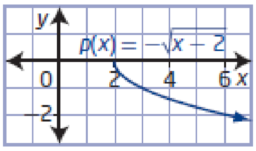
domain
$\{x \mid x \leq 0, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq -3, y \in \mathbb{R}\}$

b)



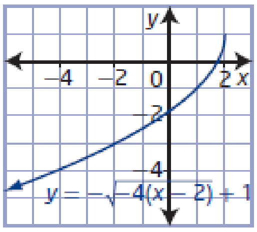
domain
$\{x \mid x \geq -1, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq 0, y \in \mathbb{R}\}$

c)



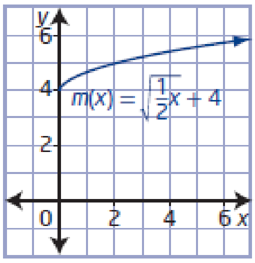
domain
$\{x \mid x \geq 2, x \in \mathbb{R}\}$,
range
$\{y \mid y \leq 0, y \in \mathbb{R}\}$

d)



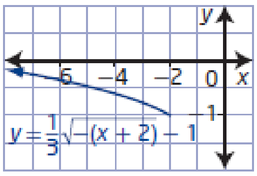
domain
$\{x \mid x \leq 2, x \in \mathbb{R}\}$,
range
$\{y \mid y \leq 1, y \in \mathbb{R}\}$

e)



domain
$\{x \mid x \geq 0, x \in \mathbb{R}\}$,
range
$\{y \mid y \geq 4, y \in \mathbb{R}\}$

f)



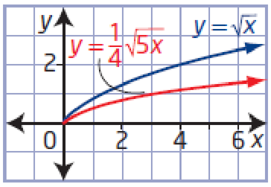
domain
$\{x \mid x \leq -2, x \in \mathbb{R}\}$
range
$\{y \mid y \geq -1, y \in \mathbb{R}\}$

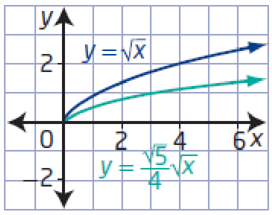
6. a) $a = \frac{1}{4} \rightarrow$ vertical stretch factor of $\frac{1}{4}$
$b = 5 \rightarrow$ horizontal stretch factor of $\frac{1}{5}$

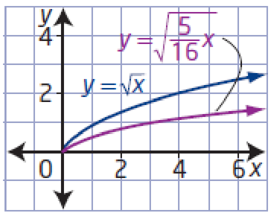
b) $y = \frac{\sqrt{5}}{4}\sqrt{x}$, $y = \sqrt{\frac{5}{16}x}$

c) $a = \frac{\sqrt{5}}{4} \rightarrow$ vertical stretch factor of $\frac{\sqrt{5}}{4}$
$b = \frac{5}{16} \rightarrow$ horizontal stretch factor of $\frac{16}{5}$

d)





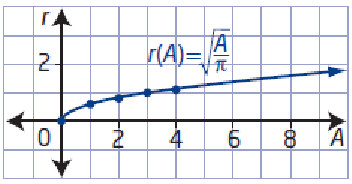


All graphs are the same.

7. a) $r(A) = \sqrt{\frac{A}{\pi}}$

b)

A	r
0	0
1	0.6
2	0.8
3	1.0
4	1.1



8. a) $b = 1.50 \rightarrow$ horizontal stretch factor of $\frac{1}{1.50}$ or $\frac{2}{3}$

b) $d \approx 1.22\sqrt{h}$ Example: I prefer the original function because the values are exact.

c) approximately 5.5 miles

9. a) domain $\{x \mid x \geq 0, x \in \mathrm{R}\}$, range $\{y \mid y \geq -13, y \in \mathrm{R}\}$

b) $h = 0 \rightarrow$ no horizontal translation
$k = 13 \rightarrow$ vertical translation down 13 units

10. a) $y = -\sqrt{x+3} + 4$ **b)** $y = \frac{1}{2}\sqrt{x+5} - 3$

c) $y = 2\sqrt{-(x-5)} - 1$ or $y = 2\sqrt{-x+5} - 1$

d) $y = -4\sqrt{-(x-4)} + 5$ or $y = -4\sqrt{-x+4} + 5$

11. Examples:

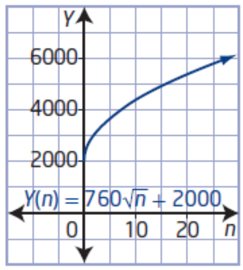
a) $y - 1 = \sqrt{x-6}$ or $y = \sqrt{x-6} + 1$

b) $y = -\sqrt{x+7} - 9$ **c)** $y = 2\sqrt{-x+4} - 3$

d) $y = -\sqrt{-(x+5)} + 8$

12. a) $a = 760 \rightarrow$ vertical stretch factor of 760
$k = 2000 \rightarrow$ vertical translation up 2000

b)



c) domain
$\{n \mid n \geq 0, n \in \mathrm{R}\}$
range
$\{Y \mid Y \geq 2000, Y \in \mathrm{R}\}$

d) The minimum yield is 2000 kg/hectare. Example: The domain and range imply that the more nitrogen added, the greater the yield without end. This is not realistic.

Square Root of a Function

Focus on...

- sketching the graph of $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- explaining strategies for graphing $y = \sqrt{f(x)}$ given the graph of $y = f(x)$
- comparing the domains and ranges of the functions $y = f(x)$ and $y = \sqrt{f(x)}$, and explaining any differences

square root of a function

- the function $y = \sqrt{f(x)}$ is the square root of the function $y = f(x)$
- $y = \sqrt{f(x)}$ is only defined for $f(x) \geq 0$ (positive y-values)

The function $y = \sqrt{2x + 1}$ represents the **square root of the function** $y = 2x + 1$.

x	$y = 2x + 1$	$y = \sqrt{2x + 1}$
0	1	1
4	9	3
12	25	5
24	49	7
⋮	⋮	⋮

$y=2x+1$

x	y
0	1
4	9
12	25
24	49

D: $\{x \mid x \in R\}$
R: $\{y \mid y \in R\}$

$y=\sqrt{2x+1}$

x	y
0	1
4	3
12	5
24	7

$2x+1 \geq 0$
$2x \geq -1$
$x \geq -\frac{1}{2}$

D: $\{x \mid x \geq -\frac{1}{2}, x \in R\}$

R: $\{y \mid y \geq 0, y \in R\}$

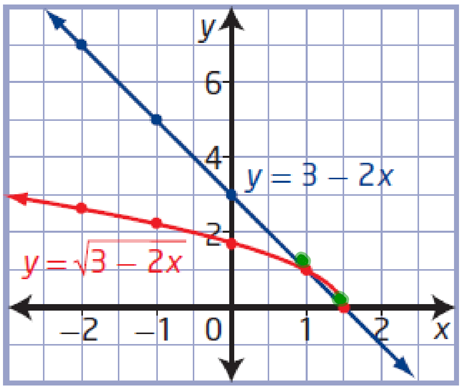
Example 1

Compare Graphs of a Linear Function and the Square Root of the Function

a) Given $f(x) = 3 - 2x$, graph the functions $y = f(x)$ and $y = \sqrt{f(x)}$.
b) Compare the two functions.

Use a table of values to graph $y = 3 - 2x$ and $y = \sqrt{3 - 2x}$.

x	$y = 3 - 2x$	$y = \sqrt{3 - 2x}$
−2	7	$\sqrt{7} = 2.64$
−1	5	$\sqrt{5} = 2.24$
0	3	$\sqrt{3} = 1.73$
1	1	1
1.5	0	0



$y = 3 - 2x$

x	y
−2	7
−1	5
0	3
1	1
1.5	0

$y = \sqrt{3-2x}$

x	y
−2	$\sqrt{7}$
−1	$\sqrt{5}$
0	$\sqrt{3}$
1	1
1.5	0

Invariant points

$3 - 2x \geq 0$
$-2x \geq -3$
$x \leq 1.5$

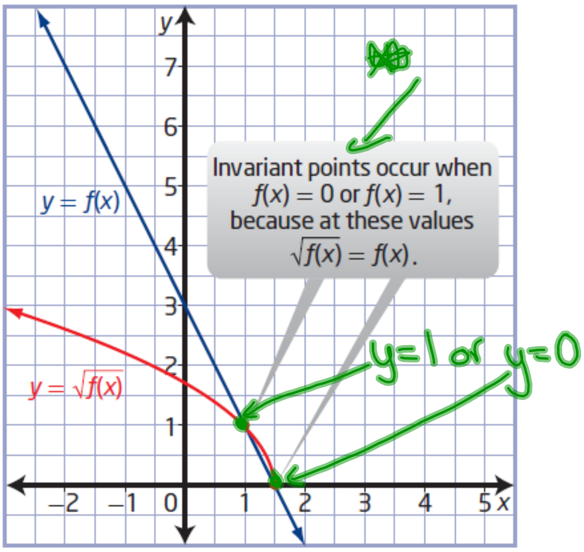
D: $\{x \mid x \in R\}$

R: $\{y \mid y \in R\}$

D: $\{x \mid x \leq 1.5, x \in R\}$

R: $\{y \mid y \geq 0, y \in R\}$

b) Compare the graphs.



Why is the graph of $y = \sqrt{f(x)}$ above the graph of $y = f(x)$ for values of y between 0 and 1? Will this always be true? yes

$y = 0.5$

$y = \sqrt{0.5} = 0.7$

For $y = f(x)$, the domain is $\{x \mid x \in R\}$ and the range is $\{y \mid y \in R\}$.
For $y = \sqrt{f(x)}$, the domain is $\{x \mid x \leq 1.5,\ x \in R\}$ and the range is $\{y \mid y \geq 0,\ y \in R\}$.

Invariant points occur at (1, 1) and (1.5, 0).

How does the domain of the graph of $y = \sqrt{f(x)}$ relate to the restrictions on the variable in the radicand? How could you determine the domain algebraically?

Relative Locations of $y = f(x)$ and $y = \sqrt{f(x)}$

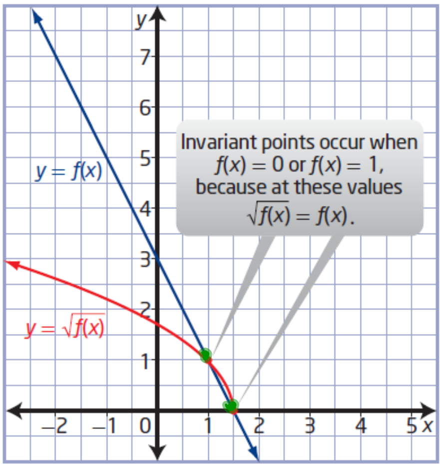
The domain of $y = \sqrt{f(x)}$ consists only of the values in the domain of $f(x)$ for which $f(x) \geq 0$.

The range of $y = \sqrt{f(x)}$ consists of the square roots of the values in the range of $y = f(x)$ for which $\sqrt{f(x)}$ is defined.

The graph of $y = \sqrt{f(x)}$ exists only where $f(x) \geq 0$. You can predict the location of $y = \sqrt{f(x)}$ relative to $y = f(x)$ using the values of $f(x)$.

Value of $f(x)$	$f(x) < 0$	Invariant $f(x) = 0$	$0 < f(x) < 1$	Invariant $f(x) = 1$	$f(x) > 1$
Relative Location of Graph of $y = \sqrt{f(x)}$	The graph of $y = \sqrt{f(x)}$ is undefined.	The graphs of $y = \sqrt{f(x)}$ and $y = f(x)$ intersect on the x-axis.	The graph of $y = \sqrt{f(x)}$ is above the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ intersects the graph of $y = f(x)$.	The graph of $y = \sqrt{f(x)}$ is below the graph of $y = f(x)$.

b) Compare the graphs.

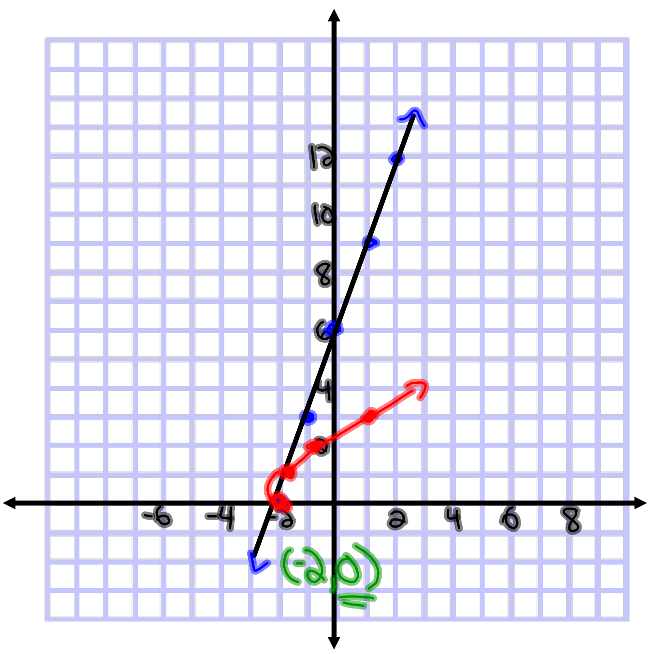


Why is the graph of $y = \sqrt{f(x)}$ above the graph of $y = f(x)$ for values of y between 0 and 1? Will this always be true?

Your Turn

a) Given $g(x) = 3x + 6$, graph the functions $y = g(x)$ and $y = \sqrt{g(x)}$.

b) Identify the domain and range of each function and any invariant points.



$g(x)=3x+6$

x	y
-3	-3
-2	0
-1	3
0	6
1	9
2	12
3	15

D: $\{x \mid x \in R\}$
R: $\{y \mid y \in R\}$

$y=\sqrt{3x+6}$

x	y
-3	undefined
-2	0
-1	$\sqrt{3}$
0	$\sqrt{6}$
1	3
2	$\sqrt{12}$
3	$\sqrt{15}$

D: $\{x \mid x \geq -2, x \in R\}$
R: $\{y \mid y \geq 0, y \in R\}$

To sketch $y=\sqrt{g(x)}$

① Locate your invariant points (where $y=0$ + $y=1$)

② Draw your square root function between the invariant points (above original)

③ Draw your square root function for $f(x)>1$ (below original)