Warm Up

Differentiate the following:

$$f(x)=\frac{-2x\tan^{-1}\sqrt{x}}{\cos^{-1}\left(\sec x^{3}\right)}$$

$$d\tan^{-1}u = \frac{1}{1+u^2}\cdot du = \frac{du}{1+u^2}$$

$$d\cos^{-1}u = \frac{-1}{\sqrt{1-u^2}}\cdot du = \frac{-du}{\sqrt{1-u^2}}$$

$$d\sec u = \sec u\tan u \cdot du$$

$$F'(x) = \frac{\cos^{-1}(\sec x^3)\left[-2x\left(\frac{1}{1+x}\cdot\frac{1}{2}x^{-1/2}\right) + \tan^{-1}\sqrt{x}\,(-2)\right] - (-2x\tan^{-1}\sqrt{x})\left[\frac{-1}{\sqrt{1-\sec^2 x^3}}\cdot \sec x^3 \tan x^3 \cdot 3x^2\right]}{\left[\cos^{-1}(\sec x^3)\right]^2}$$

Questions from Homework

② $f(x)=x\tan^{-1}x$

$$f'(x)=x\left[\frac{1}{1+x^2}\cdot 1\right]+1(\tan^{-1}x)$$

$$f'(x)=\frac{x}{1+x^2}+\tan^{-1}x$$

$$f'(1)=\frac{1}{1+(1)^2}+\tan^{-1}(1)$$

what angle has a tangent value equal to 1

$$=\frac{1}{2}+\frac{\pi}{4}$$

$$=\frac{2+\pi}{4}$$

④ $f(x)=(3\tan^{-1}x)^4$

$$f'(x)=4(3\tan^{-1}x)^3\left[3\left(\frac{1}{1+x^2}\cdot 1\right)+\cancel{(0)\tan^{-1}x}\right]$$

$$f'(x)=4(3\tan^{-1}x)^3\left[\frac{3}{1+x^2}\right]$$

$$f'(x)=\frac{12(3\tan^{-1}x)^3}{1+x^2}$$

$$f'(\sqrt{3})=\frac{12(3\tan^{-1}\sqrt{3})^3}{1+(\sqrt{3})^2}$$

what angle has a tangent value equal to $\sqrt{3}$

$$=\frac{12(3(\pi/3))^3}{1+3}$$

$$=\frac{12\pi^3}{4}$$

$$\boxed{=3\pi^3}$$

⑥ $f(x) = (x-3)(6x-x^2)^{1/2} + 9\sin^{-1}\left(\frac{x-3}{3}\right)$

$f'(x) = (x-3)\frac{1}{2}(6x-x^2)^{-1/2}(6-2x) + (6x-x^2)^{1/2} + 9\left[\frac{1}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}} \cdot \frac{1}{3}\right]$

$f'(x) = \frac{(x-3)(6-2x)}{2\sqrt{6x-x^2}} + \sqrt{6x-x^2} + \frac{3}{\sqrt{1-\left(\frac{x-3}{3}\right)^2}}$

$f'(3) = \frac{(0)(0)}{6} + 3 + 3$

$= 0 + 3 + 3$

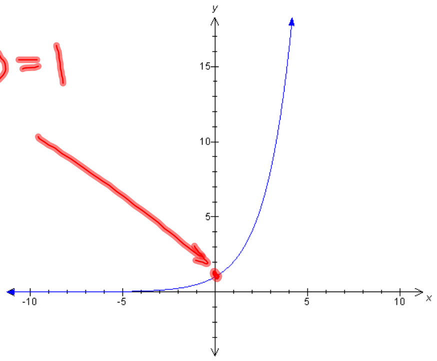
$= 6$

Differentiating Exponential Functions

What is an exponential function? $y = a^x$

$y = 2^x$ Base must be a Real number

$f(0) = 1$



when $x = -1$ $(-1, \frac{1}{2})$

$y = 2^{-1}$

$y = \frac{1}{2}$

When you do not have a rule to differentiate resort to the definition...

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Let's try and differentiate $y = a^x$ $f(x) = a^x$ $f(x+h) = a^{x+h}$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{a^{x+h} - a^x}{h}$$

$$= \lim_{h \to 0} \frac{a^x a^h - a^x}{h} = \lim_{h \to 0} \frac{a^x(a^h - 1)}{h}$$

This factor does not depend on h, therefore we can move to the front of the limit

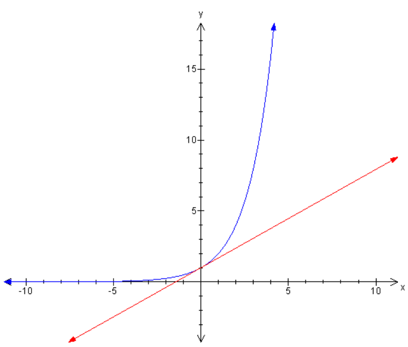
Thus we now have...

$$f'(x) = a^x \lim_{h \to 0} \frac{a^h - 1}{h}$$

What would be the value of $f'(0)$?

$$\lim_{h \to 0} \frac{a^h - 1}{h} = f'(0)$$

What would this represent in terms of slope??



We have determined that $f'(x) = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$

Same thing!

and that $\lim_{h\to 0} \frac{a^h - 1}{h} = f'(0)$

Therefore given $f(x) = a^x$, then $f'(x) = a^x f'(0)$

Here are a couple of numerical examples...

- $a = 2$; here apparently $f'(0) \approx 0.69$
- $a = 3$; here apparently $f'(0) \approx 1.10$

h	$\frac{2^h - 1}{h}$	$\frac{3^h - 1}{h}$
0.1	0.7177	1.1612
0.01	0.6956	1.1047
0.001	0.6934	1.0992
0.0001	0.6932	1.0987

There must then be some number between 2 and 3 such that

$$\lim_{h\to 0} \frac{a^h - 1}{h} = 1$$

This number turns out to be "*e*"...Euler's Number

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e^(1)
     2.718281828
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$y = 2^x$

$y' = 2^x(0.6932)$

$y = e^x$

$y' = e^x(1)$

$y' = e^x$

This leads to the following definition...

Definition of the Number e

e is the number such that $\lim_{h \to 0} \frac{e^h - 1}{h} = 1$

What does this mean geometrically?

$f(0)=1$

$f'(0)=1$

- Geometrically, this means that
  - of all the exponential functions $y = a^x$,
  - the function $f(x) = e^x$ is the one whose tangent at $(0, 1)$ has a slope $f'(0)$ that is exactly 1 .

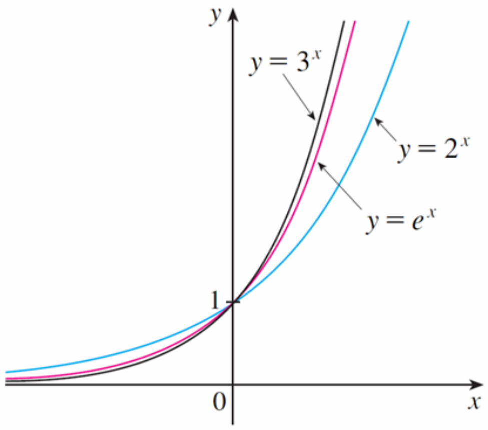

FIGURE 6

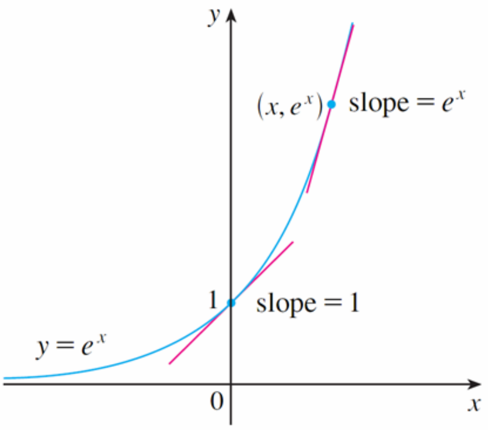

FIGURE 7

This leads to the following differentiation formula...

Derivative of the Natural Exponential Function

$$\frac{d}{dx}(e^x) = e^x$$

This is the ONLY function that is its own derivative

$$f(x) = e^x$$
$$f'(x) = e^x$$

In General...

$$\frac{d(e^u)}{dx} = e^u \bullet du$$

Differentiating Exponential Functions

$y = e^{3x^7}$

$y' = e^{3x^7} \cdot 21x^6$

$y' = 21x^6 e^{3x^7}$

$y = e^{\sin x}$

$y' = e^{\sin x} \cdot \cos x \cdot 1$

$y' = \cos x \, e^{\sin x}$

$y = (x^2)(e^x)$

$y' = 2xe^x + x^2e^x$

$y' = xe^x(2+x)$

$y = e^{\cot x^3}$

$y' = e^{\cot x^3} \cdot -\csc^2 x^3 \cdot 3x^2$

$y' = -3x^2 \csc^2 x^3 e^{\cot x^3}$

Practice Exercises

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#4, 5, 6, 8, 9, 10,

Bonus:

Give that $y = \cos^{-1}(\cos^{-1} x)$, prove that

$$\frac{dy}{dx} = \frac{1}{\sin y\sqrt{1-x^2}}$$