6.4

Factored Form of a Quadratic Function

$y = a(x-r)(x-s)$

GOAL

Relate the factors of a quadratic function to the characteristics of its graph.

$y = x^2 + 7x + 12$

$4 \times 3 = 12$

$4 + 3 = 7$

$y = (x+4)(x+3)$

Standard Form:

$y = ax^2 + bx + c$

Reflecting

- When a quadratic function is in factored form, what information can you get from the values of a, r, and s?
- What does the product $(a)(r)(s)$ tell you about the graph of the function? = y - intercept

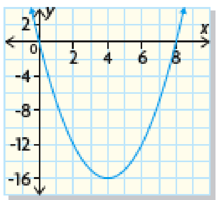
$c = (a)(r)(s)$

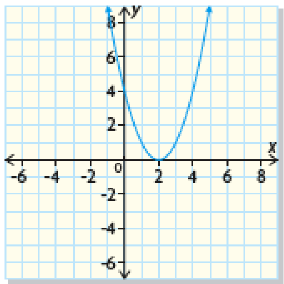
Communication Tip
A quadratic function is in factored form when it is written in the form $y = a(x - r)(x - s)$

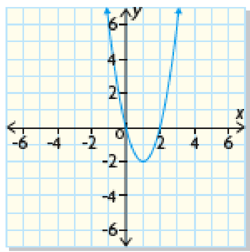
a = stretch factor
r = x- intercept
s = x- intercept

Match the functions with the parabolas.

$y = x^2 - 8x$ $y = (x)(x - 8)$	$y = x^2 - 4x + 4$ $y = (x - 2)(x - 2)$	$y = 2x^2 - 4x$ $y = 2(x)(x - 2)$







H. How are the x-intercepts of the parabola related to the factors of your function?

I. Explain why having a quadratic function in factored form is useful when graphing the parabola.

Answers

H. I can determine the x-intercepts by setting each factor equal to zero in turn.

I. Factored form makes it easier to see where the x-intercepts are. Some students may add the following: I can also use the average of the x-intercepts to determine the axis of symmetry.

APPLY the Math

EXAMPLE 1 Graphing a quadratic function given in standard form

standard: $a = 2 \rightarrow$ opens up (stretch factor of 2)
$b = 14$
$c = 12 \rightarrow$ y-int $= 12$
$(0, 12)$

Sketch the graph of the quadratic function:

$f(x) = 2x^2 + 14x + 12$

State the domain and range of the function.

$\{x \mid x \in R\}$

Arvin's Solution

$f(x) = 2x^2 + 14x + 12$ (Standard)
The coefficient of x^2 is $+2$, so the parabola opens upward.

The parabola opens upward when a is positive in the standard form of the function.

common factor of 2

$f(x) = 2(x^2 + 7x + 6)$
$f(x) = \underbrace{2(x + 1)(x + 6)}_{\text{factored}}$

$6 \times 1 = 6$
$6 + 1 = 7$

I factored the expression on the right side so that I could determine the **zeros** of the function.

zero
In a function, a value of the variable that makes the value of the function equal to zero.

Zeros: (x-intercepts)
$0 = 2(x + 1)(x + 6)$
$x + 1 = 0$ or $x + 6 = 0$
$x = -1$ $x = -6$
The x-intercepts are $x = -1$ and $x = -6$. or $(-1, 0)$ and $(-6, 0)$

To determine the zeros, I set $f(x)$ equal to zero. I knew that a product is zero only when one or more of its factors are zero, so I set each factor equal to zero and solved each equation.

The values of x at the zeros of the function are also the x-intercepts.

y-intercept:
$f(0) = 2(0 + 1)(0 + 6)$
$f(0) = 2(1)(6)$
$f(0) = 12$
The y-intercept is 12.

I knew that the y-intercept is 12 from the standard form of the quadratic function. However, I decided to verify that my factoring was correct.

I noticed that this value can be obtained by multiplying the values of a, r, and s from the factored form of the function: $f(x) = a(x - r)(x - s)$

Axis of symmetry:

$$x = \frac{-6 + (-1)}{2}$$

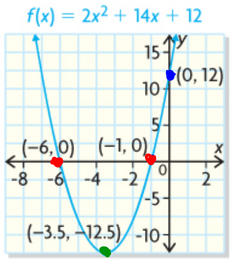
$x = -3.5$

The axis of symmetry passes through the midpoint of the line segment that joins the x-intercepts. I calculated the mean of the two x-intercepts to determine the equation of the axis of symmetry.

$y = 2(x+1)(x+6)$

$f(x) = 2(x + 1)(x + 6)$
$f(-3.5) = 2(-3.5 + 1)(-3.5 + 6)$
$f(-3.5) = 2(-2.5)(2.5)$
$f(-3.5) = -12.5$
The vertex of the parabola is $(-3.5, -12.5)$. (vertex)

The vertex lies on the axis of symmetry, so its x-coordinate is -3.5. I substituted -3.5 into the equation to determine the y-coordinate of the vertex.



I plotted the x-intercepts, y-intercept, and vertex and then joined these points with a smooth curve.

Domain and range:
$\{(x, y) \mid x \in R, y \geq -12.5, y \in R\}$

The only restriction on the variables is that y must be greater than or equal to -12.5, the minimum value of the function.

EXAMPLE 2 Using a partial factoring strategy to sketch the graph of a quadratic function

Sketch the graph of the following quadratic function:

$$f(x) = -x^2 + 6x + 10$$

State the domain and range of the function.

Elliot's Solution

$$f(x) = -x^2 + 6x + 10$$
$$f(x) = -x(x - 6) + 10$$

I couldn't identify two integers with a product of 10 and a sum of 6, so I couldn't factor the expression. I decided to remove a partial factor of $-x$ from the first two terms. I did this so that I could determine the x-coordinates of the points that have 10 as their y-coordinate.

$-x = 0$ $\quad$ $x - 6 = 0$
$x = 0$ $\quad$ $x = 6$
$f(0) = 10$ $\quad$ $f(6) = 10$

The points (0, 10) and (6, 10) belong to the given quadratic function.

I determined two points in the function by setting each partial factor equal to zero.

When either factor is zero, the product of the factors is zero, so the value of the function is 10.

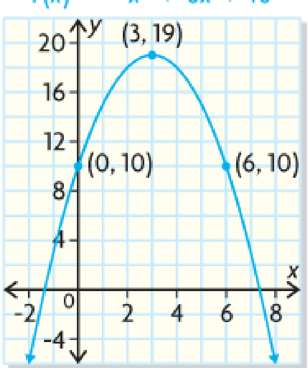
$$x = \frac{0 + 6}{2}$$
$$x = 3$$

Because (0, 10) and (6, 10) have the same y-coordinate, they are the same horizontal distance from the axis of symmetry. I determined the equation of the axis of symmetry by calculating the mean of the x-coordinates of these two points.

$$f(3) = -(3)^2 + 6(3) + 10$$
$$f(3) = -9 + 18 + 10$$
$$f(3) = 19$$

I determined the y-coordinate of the vertex.

The vertex is (3, 19).



The coefficient of the x^2 term is negative, so the parabola opens downward.

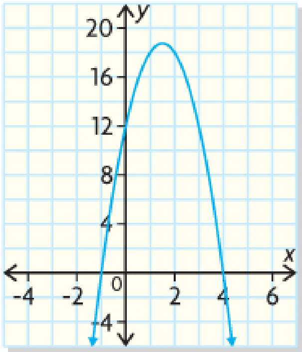
I used the vertex, as well as (0, 10) and (6, 10), to sketch the parabola.

Domain: $\{x \mid x \in \mathrm{R}\}$
Range: $\{y \mid y \leq 19, y \in \mathrm{R}\}$

The only restriction on the variables is that y must be less than or equal to 19, the maximum value of the function.

EXAMPLE 3 Determining the equation of a quadratic function, given its graph

Determine the function that defines this parabola. Write the function in standard form.



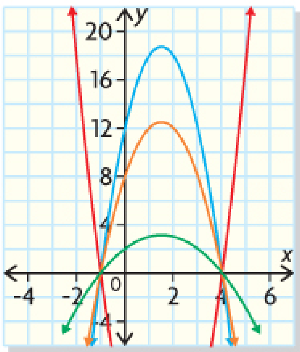
Indira's Solution

The x-intercepts are $x = -1$ and $x = 4$. The zeros of the function occur when x has values of -1 and 4.

$y = a(x - r)(x - s)$
$y = a[x - (-1)][x - (4)]$
$y = a(x + 1)(x - 4)$

The graph is a parabola, so it is defined by a quadratic function.

I located the x-intercepts and used them to determine the zeros of the function. I wrote the factored form of the quadratic function, substituting -1 and 4 for r and s.



I knew that there are infinitely many quadratic functions that have these two zeros, depending on the value of a. I had to determine the value of a for the function that defines the blue graph.

The y-intercept is 12.

$$y = a(x + 1)(x - 4)$$
$$(12) = a[(0) + 1][(0) - 4]$$
$$12 = a(1)(-4)$$
$$12 = -4a$$
$$-3 = a$$

From the graph, I determined the coordinates of the y-intercept.

Because these coordinates are integers, I decided to use the y-intercept to solve for a.

In factored form, the quadratic function is $y = -3(x + 1)(x - 4)$

I substituted the value of a into my equation.

In standard form, the quadratic function is

$y = -3(x^2 - 3x - 4)$
$y = -3x^2 + 9x + 12$

My equation seems reasonable, because it defines a graph with a y-intercept of 12 and a parabola that opens downward.

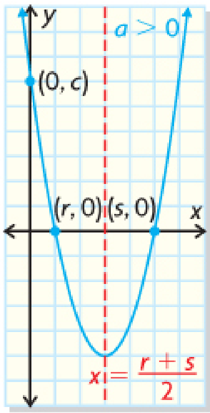
In Summary

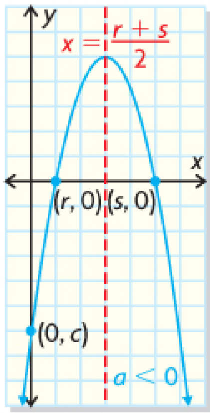
Key Ideas

- When a quadratic function is written in factored form
  $$y = a(x - r)(x - s)$$
  each factor can be used to determine a zero of the function by setting each factor equal to zero and solving.
- The zeros of a quadratic function correspond to the x-intercepts of the parabola that is defined by the function.
- If a parabola has one or two x-intercepts, the equation of the parabola can be written in factored form using the x-intercept(s) and the coordinates of one other point on the parabola.
- Quadratic functions without any zeros cannot be written in factored form.

Need to Know

- A quadratic function that is written in the form
  $$f(x) = a(x - r)(x - s)$$
  has the following characteristics:
  - The x-intercepts of the graph of the function are $x = r$ and $x = s$.
  - The linear equation of the axis of symmetry is $x = \frac{r + s}{2}$.
  - The y-intercept, c, is $c = a \cdot r \cdot s$.





- If a quadratic function has only one x-intercept, the factored form can be written as follows:
  $$f(x) = a(x - r)(x - r)$$
  $$f(x) = a(x - r)^2$$

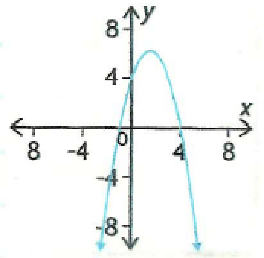
<u>Assignment:</u> pages 309 - 311
Questions 1, 3, 4(a-d), 10a, 11ad

*** Please change 1e to -(x - 1)(x + 4) and**
1f to -(x + 1)(x - 4)

SOLUTIONS ⇒ 6.4 Factored Form of a Quadratic Function.

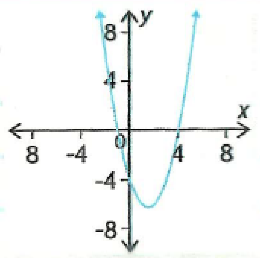
1. Match each quadratic function with its corresponding parabola.

i)

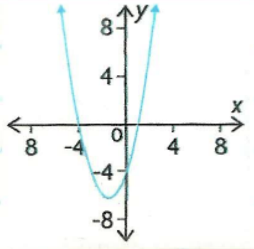


Match: f) $f(x) = (x+1)(4-x)$
$= (x+1)[-(-4+x)]$
$f(x) = -(x+1)(x-4)$

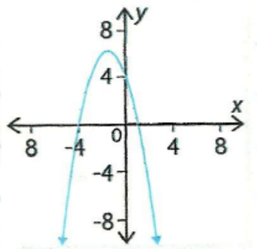
ii)



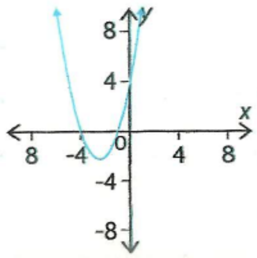
Match: b) $f(x) = (x+1)(x-4)$

iii) 

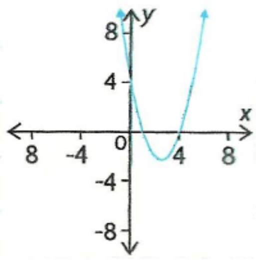
Match: a) $f(x) = (x-1)(x+4)$

iv) 

Match: e) $f(x) = (1-x)(x+4)$
$= -[(-1+x)](x+4)$
$= -(x-1)(x+4)$

v) 

Match: c) $f(x) = (x+1)(x+4)$

vi) 

Match: d) $f(x) = (x-1)(x-4)$

3. A quadratic function has an equation that can be written in the form $f(x) = a(x-r)(x-s)$. The graph of the function has x-intercepts $x=-2$ and $x=4$ and passes through point $(5,7)$. Write the equation of the quadratic function.

$f(x) = a(x-r)(x-s)$ $\quad$ $r=-2$, $s=4$, $(\underset{x}{5},\underset{y}{7})$

$\Rightarrow y = a(x--2)(x-4)$

$\Rightarrow y = a(x+2)(x-4)$

$$\left.\begin{aligned} 7 &= a(5+2)(5-4) \\ 7 &= a(7)(1) \\ \frac{7}{7} &= \frac{\cancel{7}a}{\cancel{7}} \\ 1 &= a \end{aligned}\right\}$$

To find "a"
Substitute $x=5$ & $y=7$.

$\Rightarrow y = 1(x+2)(x-4)$

{a-d} 4. For each quadratic function, determine the x-intercepts, the y-intercept, the equation of the axis of symmetry, and the coordinates of the vertex of the graph.

a) $f(x) = (x-1)(x+1)$

x-intercepts: $x = r$ and $x = s$
$x = 1$ $\quad x = -1$

y-intercept: $c = a \cdot r \cdot s$
$c = (1)(1)(-1)$
$c = -1$

Axis of Symmetry: $x = \frac{r+s}{2}$
$x = \frac{1 + -1}{2}$
$x = \frac{0}{2}$
$x = 0$

Vertex: (x=0)
$y = (x-1)(x+1)$
$y = (0-1)(0+1)$
$y = (-1)(1)$
$y = -1$

$(0, -1)$

b) $f(x) = (x+2)(x+2)$

x-intercepts: $x = r$ and $x = s$
$x = -2$ $x = -2$ (SAME)

y-intercept: $c = a \cdot r \cdot s$
$c = (1)(-2)(-2)$
$c = 4$

Axis of Symmetry: $x = \frac{r+s}{2}$
$x = \frac{-2+-2}{2}$
$x = \frac{-4}{2}$
$x = -2$

Vertex: (x=-2)
$y = (x+2)(x+2)$
$y = (-2+2)(-2+2)$
$y = (0)(0)$
$y = 0$
(-2, 0)

c) $f(x) = (x-3)(x-3)$

x-intercepts: $x = r$ and $x = s$
$x = 3$ $x = 3$ (SAME)

y-intercept: $c = a \cdot r \cdot s$
$c = (1)(3)(3)$
$c = 9$

Axis of Symmetry: $x = \frac{r+s}{2}$
$x = \frac{3+3}{2}$
$x = \frac{6}{2}$
$x = 3$

Vertex: $y = (x-3)(x-3)$ (3,0)
($x=3$) $y = (3-3)(3-3)$
$y = (0)(0)$
$y = 0$

d) $f(x) = -2(x-2)(x+1)$

x-intercepts: $x = r$ and $x = s$
$x = 2$ $x = -1$

y-intercept: $c = a \cdot r \cdot s$
$c = (-2)(2)(-1)$
$c = 4$

Axis of Symmetry: $x = \frac{r+s}{2}$
$x = \frac{2+-1}{2}$
$x = \frac{1}{2}$
$x = 0.5$

Vertex: $(x = 0.5)$
$y = -2(x-2)(x+1)$
$= -2(0.5-2)(0.5+1)$
$= -2(-1.5)(1.5)$
$= 4.5$

$(0.5, 4.5)$

{a}

10. For each quadratic function below

i) use partial factoring to determine two points that are the same distance from the axis of symmetry.

ii) determine the coordinates of the vertex.

iii) sketch the graph.

a) i) $f(x)=x^2+4x-6$

$f(x)=x(x+4)-6$

$x=0$ or $x+4=0$

$x=-4$

$f(0)=-6$ $f(-4)=-6$

Two Points: $(0,-6)$ & $(-4,-6)$

ii) To locate the vertex:

$x=\frac{0+-4}{2}$

$x=-\frac{4}{2}$

$x=-2$

$f(-2)=(-2)^2+4(-2)-6$

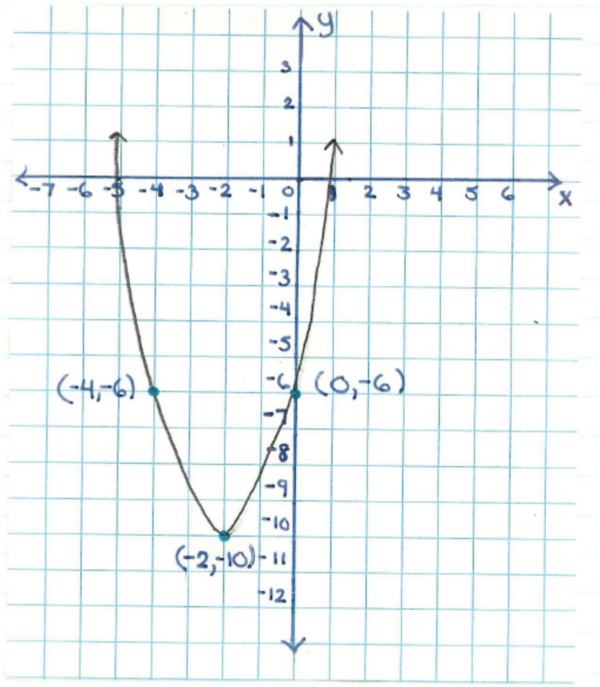
$=4-8-6$

$=-4-6$

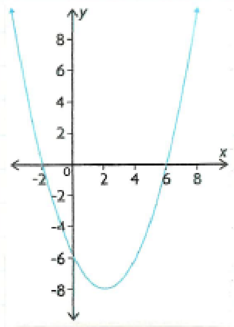
$=-10$

Vertex: $(-2,-10)$

iii)



11. Determine the equation of the quadratic function that defines each parabola.
{a, d}

a) 

x-intercepts: $x = \underset{r}{-2}$ & $x = \underset{s}{6}$

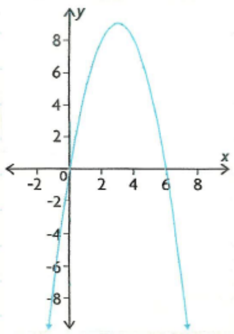
Vertex: $(\underset{x}{2}, \underset{y}{-8})$

$y = a(x-r)(x-s)$
$y = a(x--2)(x-6)$
$y = a(x+2)(x-6)$

To find a:
$-8 = a(2+2)(2-6)$
$-8 = a(4)(-4)$
$\frac{-8}{-16} = \frac{-16a}{-16}$
$\frac{1}{2} = a$

$\Rightarrow \quad y = \frac{1}{2}(x+2)(x-6)$
$y = \left(\frac{1}{2}x+1\right)(x-6)$
$y = \frac{1}{2}x^2 - 3x + 1x - 6$
$y = \frac{1}{2}x^2 - 2x - 6$

d)



x-intercepts: $x = 0$ & $x = 6$

$\underset{r}{0}$, $\underset{s}{6}$

Vertex: $(3, 9)$

$\underset{x}{3}$, $\underset{y}{9}$

$y = a(x-r)(x-s)$
$y = a(x-0)(x-6)$
$y = a(x)(x-6)$

To find a:
$9 = a(3)(3-6)$
$9 = a(3)(-3)$
$\frac{9}{-9} = \frac{-\cancel{9}a}{-\cancel{9}}$
$-1 = a$

$\Rightarrow y = -1(x)(x-6)$
$y = -1x(x-6)$
$y = -1x^2 + 6x$

Attachments

7s4e1 final.mp4

7s4e2 final.mp4

7s4e3 final.mp4

7s4e4 final.mp4

fm7s4-p11.tns

FM11-7s4.gsp