Questions from Homework

④ $f(x) = \sqrt{1+x^3} = (1+x^3)^{1/2}$

$$f'(x) = \frac{1}{2}(1+x^3)^{-1/2}(3x^2)$$

$$= \frac{3x^2}{2(1+x^3)^{1/2}}$$

$$f''(x) = \frac{2(1+x^3)^{1/2}(6x) - 3x^2(1)(1+x^3)^{-1/2}(3x^2)}{\left[2(1+x^3)^{1/2}\right]^2}$$

$$= \frac{12x(1+x^3)^{1/2} - 9x^4(1+x^3)^{-1/2}}{4(1+x^3)}$$

$$= \frac{3x(1+x^3)^{-1/2}\left[4(1+x^3) - 3x^3\right]}{4(1+x^3)}$$

$$= \frac{3x(1+x^3)^{-1/2}(4+x^3)}{4(1+x^3)}$$

$$= \frac{3x(4+x^3)}{4(1+x^3)(1+x^3)^{1/2}} = \frac{3x(4+x^3)}{4(1+x^3)^{3/2}}$$

$$f''(2) = \frac{3(2)\left[4+(2)^3\right]}{4\sqrt{(1+(2)^3)^3}}$$

$$= \frac{6(12)}{4\sqrt{729}}$$

$$= \frac{72}{4(27)}$$

$$= \frac{72}{108}$$

$$= \boxed{\frac{2}{3}}$$

⑧ Quadratic Function

$f(3) = 33$ $\quad \boxed{f(x) = 4x^2 - 2x + 3}$

$f'(3) = 22$ $\quad f'(x) = 8x - 2$

$f''(3) = 8$ $\quad f''(x) = 8$

⑤ $g(x) = \dfrac{1}{\sqrt{3x+4}} = \dfrac{1}{(3x+4)^{1/2}} = (3x+4)^{-1/2}$

$g'(x) = -\frac{1}{2}(3x+4)^{-3/2}(3) = \frac{-3}{2}(3x+4)^{-3/2}$

$g''(x) = \frac{9}{4}(3x+4)^{-5/2}(3) = \frac{27}{4}(3x+4)^{-5/2}$

$g'''(x) = \frac{-135}{8}(3x+4)^{-7/2}(3)$

$g'''(x) = \dfrac{-405}{8(3x+4)^{7/2}}$

$g'''(x) = \dfrac{-405}{8\sqrt{(3x+4)^7}}$

$g'''(4) = \dfrac{-405}{8\sqrt{(3(4)+4)^7}}$

$= \dfrac{-405}{8\sqrt{(16)^7}}$

$\boxed{= \dfrac{-405}{131072}}$

Implicit Differentiation

So far we have described functions by expressing one variable *explicitly* in terms of another variable: for example,

$y = x^2$

$y' = 2x$

$\frac{dy}{dx} = 2x$

- Sometimes an equation only <u>implicitly</u> defines y as a function (or functions) of x.
- Examples
  - $x^2 + y^2 = 25$
  - $x^3 + y^3 = 6xy$

$x^2 + y^2 = 25$

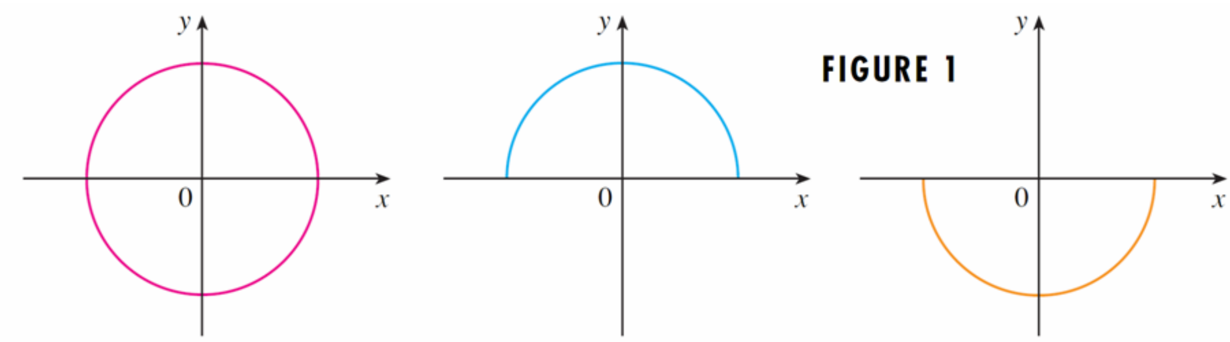
$y^2 = 25 - x^2$

$y = \pm\sqrt{25 - x^2}$

- The first equation could easily be rearranged for $y = \ldots$

$$y = \pm\sqrt{25 - x^2}$$ ← Actually gives two functions

FIGURE 1



(a) $x^2 + y^2 = 25$ (b) $f(x) = \sqrt{25 - x^2}$ (c) $g(x) = -\sqrt{25 - x^2}$

Implicit Differentiation

- There is a way called *implicit differentiation* to find dy/dx without solving for y :
  - First differentiate both sides of the equation with respect to x ;
  - Then solve the resulting equation for y' . or $\frac{dy}{dx}$
- We will always assume that the given equation does indeed define y as a differentiable function of x .

Example

- For the circle $x^2 + y^2 = 25$, find
  a) dy/dx or y' or (Slope of the tangent)
  b) an equation of the tangent at the point (3, 4) .

Solution:

Start by differentiating both sides of the equation:

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}(25)$$

$$\frac{d}{dx}(x^2) + \frac{d}{dx}(y^2) = 0$$

Remembering that y is a function of x and using the Chain Rule, we have

$$2x + 2y(y') = 0$$

Thus...

$$2x + 2y\frac{dy}{dx} = 0$$

$$y'$$

Solving for $\frac{dy}{dx}$ ...

$$2yy' = -2x$$

$$y' = \frac{-2x}{2y}$$

$$y' = \frac{-x}{y} = \frac{-3}{4} = m$$

Therefore at the point (3,4) the equation of the tangent would be...

$$y - 4 = -\tfrac{3}{4}(x - 3) \quad \text{or} \quad \boxed{\cancel{3x + 4y = 25}}$$

$$y - 4 = -\frac{3}{4}x + \frac{9}{4}$$

$$4y - 16 = -3x + 9$$

$$\boxed{3x + 4y - 25 = 0}$$

Same Example Revisited

- Since it is easy to solve this equation for y, we
  - do so, and then
  - find the equation of the tangent line by earlier methods, and then
  - compare the result with our preceding answer:

Solution

- Solving the equation gives $y = \pm\sqrt{25 - x^2}$ as before.
- The point (3, 4) lies on the <u>upper</u> semicircle $y = \sqrt{25 - x^2}$ and so we consider the function $f(x) = \sqrt{25 - x^2}$

Differentiate *f*:

$$y = \sqrt{25 - x^2} = (25 - x^2)^{1/2}$$

$$y' = \frac{1}{2}(25 - x^2)^{-1/2}(-2x)$$

$$y' = \frac{-x}{\sqrt{25 - x^2}} = \frac{-(3)}{\sqrt{25 - (3)^2}} = \frac{-3}{4}$$

Equation:

$$y - 4 = \frac{-3}{4}(x - 3)$$

$$y - 4 = \frac{-3x}{4} + \frac{9}{4}$$

$$4y - 16 = -3x + 9$$

$$\boxed{3x + 4y + 25 = 0}$$

Solution (cont'd)

- So $f'(3) = -\dfrac{3}{\sqrt{25-3^2}} = -\dfrac{3}{4}$,

  leading to the same equation

  $$3x + 4y = 25$$

  for the tangent that we obtained earlier.
- Note that although this problem could be done both ways, implicit differentiation was easier!

Sometimes **Implicit Differentiation** **is not only the easiest way, it's the *only* way**

Example:

Given $$x^3 + y^3 = \boxed{6x}\boxed{y}$$

first second

Find $\frac{dy}{dx}$

$$3x^2 + 3y^2y' = 6xy' + 6y$$

Product Rule

$$3y^2y' - 6xy' = -3x^2 + 6y$$

Factor

$$y'(3y^2 - 6x) = -3x^2 + 6y$$

$$y' = \frac{-3x^2 + 6y}{3y^2 - 6x}$$

$$y' = \frac{-\cancel{3}(x^2 - 2y)}{\cancel{3}(y^2 - 2x)}$$

$$\boxed{y' = -\frac{x^2 - 2y}{y^2 - 2x}}$$

Product Rule

Find $\frac{dy}{dx}$ $$2x^5 + x^4y + y^5 = 36$$

$$10x^4 + x^4y' + 4x^3y + 5y^4y' = 0$$

$$x^4y' + 5y^4y' = -10x^4 - 4x^3y$$

Factor $$y'(x^4 + 5y^4) = -10x^4 - 4x^3y$$

$$y' = \frac{-10x^4 - 4x^3y}{x^4 + 5y^4}$$

$$\boxed{y' = -\frac{10x^4 + 4x^3y}{x^4 + 5y^4}}$$

Homework

① h) $\frac{2x}{x+y} = y$

$$\frac{2(x+y) - 2x(1+y')}{(x+y)^2} = y'$$

$$\frac{\cancel{2x} + 2y - \cancel{2x} - 2xy'}{(x+y)^2} = y'$$

$$\frac{2y - 2xy'}{(x+y)^2} = y'$$

$$2y - 2xy' = y'(x+y)^2$$

$$2y = y'(x+y)^2 + 2xy'$$

Factor

$$2y = y'\left[(x+y)^2 + 2x\right]$$

$$\boxed{\frac{2y}{(x+y)^2 + 2x} = y'}$$

g) $\sqrt{x} + \sqrt{y} = 1$

$$\frac{1}{2}x^{-1/2} + \frac{1}{2}y^{-1/2}y' = 0$$

$$\frac{1}{2\sqrt{x}} + \frac{y'}{2\sqrt{y}} = 0$$

$$\frac{y'}{2\sqrt{y}} = -\frac{1}{2\sqrt{x}}$$

$$y' = -\frac{\cancel{2}\sqrt{y}}{\cancel{2}\sqrt{x}} = \frac{-\sqrt{y}}{\sqrt{x}}$$

③ e) $\sqrt{x+y} + \sqrt{xy} = 4 \qquad (2,2)$

$$(x+y)^{1/2} + (xy)^{1/2} = 4$$

Product Rule

$$\frac{1}{2}(x+y)^{-1/2}(1+y') + \frac{1}{2}(xy)^{-1/2}(xy'+y) = 0$$

$$\frac{1+y'}{2\sqrt{x+y}} + \frac{xy'+y}{2\sqrt{xy}} = 0$$

LCD: $2\sqrt{x+y}\sqrt{xy}$

$$\sqrt{xy}(1+y') + \sqrt{x+y}(xy'+y) = 0$$

$$\sqrt{xy} + y'\sqrt{xy} + xy'\sqrt{x+y} + y\sqrt{x+y} = 0$$

$$y'(\sqrt{xy} + x\sqrt{x+y}) = -\sqrt{xy} - y\sqrt{x+y}$$

$$y' = -\frac{(\sqrt{xy} + y\sqrt{x+y})}{\sqrt{xy} + x\sqrt{x+y}}$$

$$y' = -\frac{\sqrt{(2)(2)} + (2)\sqrt{(2)+(2)}}{\sqrt{(2)(2)} + (2)\sqrt{(2)+(2)}}$$

$$y' = -\frac{6}{6} = -1$$

④ a) $9x^2 + 4y^2 = 36 \qquad (\sqrt{2}, \frac{3}{2}\sqrt{2})$

$$18x + 8yy' = 0$$

$$8yy' = -18x$$

$$y' = \frac{-18x}{8y} = \frac{-9x}{4y} = \frac{-9\sqrt{2}}{4(\frac{3}{2}\sqrt{2})} = \frac{-9\cancel{\sqrt{2}}}{6\cancel{\sqrt{2}}} =$$

② g) $y = \dfrac{\csc x}{1+\cot x}$

$$y' = \frac{(1+\cot x)(-\csc x\cot x\,(1)) - \csc x(-\csc^2 x\,(1))}{(1+\cot x)^2}$$

$$y' = \frac{-\csc x\cot x - \csc x\cot^2 x + \csc^3 x}{(1+\cot x)^2}$$

$$y' = \frac{-\csc x(\cot x + \underbrace{\cot^2 x - \csc^2 x}_{-1})}{1+\cot x}$$

$$\boxed{y' = \frac{-\csc x(\cot x - 1)}{(1+\cot x)^2}}$$

#3

$$y = \sqrt[7]{2x^2 + \sqrt{x^7 - 8x\sqrt{3-x^3}}}$$

$$= \left[2x^2 + \left(x^7 - 8x(3-x^3)^{1/2}\right)^{1/2}\right]^{1/7}$$

$$y' = \frac{1}{7}\left[2x^2 + \left(x^7 - 8x(3-x^3)^{1/2}\right)^{1/2}\right]^{-6/7}\left[4x + \frac{1}{2}\left(x^7 - 8x(3-x^3)^{1/2}\right)^{-1/2}\left(7x^6 - \left[8x\left(\frac{1}{2}\right)(3-x^3)^{-1/2}(-3x^2) - 8(3-x^3)^{1/2}\right]\right)\right]$$

④ $y = \frac{(2x+3)^3}{\sqrt{4x-7}} = \frac{(2x+3)^3}{(4x-7)^{1/2}}$

$x = 2$ $(2, 343)$
$y = 343$ ← point

$y = \frac{(2(2)+3)^3}{\sqrt{4(2)-7}}$

$= \frac{7^3}{\sqrt{1}}$

$= 343$

$y' = \frac{(4x-7)^{1/2}(3)(2x+3)^2(2) - (2x+3)^3\left(\frac{1}{2}\right)(4x-7)^{-1/2}(4)}{4x-7}$

$y'(2) = \frac{(1)(3)(49)(2) - (343)(1/2)(1)(4)}{1}$

$= 294 - 686$

$= -392$ ← Slope

$y - y_1 = m(x - x_1)$

$y - 343 = -392(x-2)$

$y - 343 = -392x + 784$

$\boxed{392x + y - 1127 = 0}$