Combining Functions

- This is summarized in the following table:

Algebra of Functions Let f and g be functions with domains A and B. Then the functions $f+g$, $f-g$, fg, and f/g are defined as follows:

$(f+g)(x) = f(x) + g(x)$	domain $= A \cap B$
$(f-g)(x) = f(x) - g(x)$	domain $= A \cap B$
$(fg)(x) = f(x)g(x)$	domain $= A \cap B$
$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$	domain $= \{x \in A \cap B \mid g(x) \neq 0\}$

"Intersection" or Overlap

Example

- If $f(x)=\sqrt{x}$ and $g(x)=\sqrt{4-x^2}$, find the functions $f+g$, $f-g$, fg, and f/g.

**Also examine the domain of each of these new functions

$f(x)=\sqrt{x}$ $\quad x\geq 0$

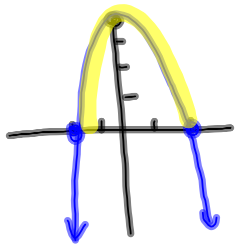
D: $\{x \mid x\geq 0, x\in\mathbb{R}\}$

$g(x)=\sqrt{4-x^2}$ $\quad 4-x^2\geq 0$

$(2-x)(2+x)\geq 0$

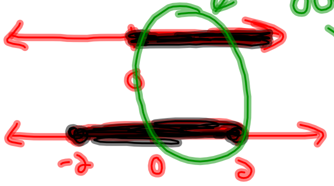
Sketch $g(x)=4-x^2$

- parabola
- roots: $x=\pm 2$
- opens down



D: $\{x \mid -2\leq x\leq 2, x\in\mathbb{R}\}$

Common domain



a) $(f+g)(x)=\sqrt{x}+\sqrt{4-x^2}$ $\quad$ D: $\{x \mid 0\leq x\leq 2, x\in\mathbb{R}\}$

b) $(f-g)(x)=\sqrt{x}-\sqrt{4-x^2}$ $\quad$ D: $\{x \mid 0\leq x\leq 2, x\in\mathbb{R}\}$

c) $(f\cdot g)(x)=(\sqrt{x})(\sqrt{4-x^2})$ $\quad$ D: $\{x \mid 0\leq x\leq 2, x\in\mathbb{R}\}$

$=\sqrt{4x-x^3}$

d) $\left(\frac{f}{g}\right)(x)=\frac{(\sqrt{x})}{(\sqrt{4-x^2})}\frac{(\sqrt{4-x^2})}{(\sqrt{4-x^2})}$ Rationalize $\quad$ D: $\{x \mid 0\leq x< 2, x\in\mathbb{R}\}$

$=\frac{\sqrt{4x-x^3}}{4-x^2}$

$x\neq\pm 2$

Compositions of Functions

(combining functions in a different way)

When the input in a function is another function, the result is called a **composite function.** If

$f(x) = 3x + 2$ and $g(x) = 4x - 5$

then $f[g(x)]$ is a composite function. The statement $f[g(x)]$ is read " f of g of x" or "the composition of f with g." $f[g(x)]$ can also be written as

$$(f \circ g)(x) \quad \text{or} \quad f \circ g(x) \quad = f[g(x)]$$

The symbol between f and g is a small open circle. When replacing one function with another, be very careful to get the order correct because compositions of functions are not necessarily commutative (as you'll see).

Example 1

Evaluate a Composite Function

If $f(x) = 4x$, $g(x) = x + 6$, and $h(x) = x^2$, determine each value.

a) $f(g(3))$
b) $g(h(-2))$
c) $h(h(2))$

Method 1: Determine the Value of the Inner Function and Then Substitute

a) $f(g(3))$
$g(3) = (3) + 6 = 9$
$f(9) = 4(9) = \boxed{36}$

b) $g(h(-2))$
$h(-2) = (-2)^2 = 4$
$g(4) = 4 + 6 = \boxed{10}$

c) $h(h(2))$
$h(2) = (2)^2 = 4$
$h(4) = (4)^2 = \boxed{16}$

Example 2

If $f(x) = 3x^2 + 2x + 1$ and $g(x) = 4x - 5$, find each of the following:

1. $f[g(x)]$
2. $g[f(x)]$

① $f(4x-5) = 3(4x-5)^2 + 2(4x-5) + 1$

$= 3(16x^2 - 40x + 25) + 8x - 10 + 1$

$= 48x^2 - 120x + 75 + 8x - 10 + 1$

$\boxed{= 48x^2 - 112x + 66}$

$= 2(24x^2 - 56x + 33)$

② $g(3x^2+2x+1) = 4(3x^2+2x+1) - 5$

$= 12x^2 + 8x + 4 - 5$

$\boxed{= 12x^2 + 8x - 1}$

Key Ideas

- Two functions, $f(x)$ and $g(x)$, can be combined using composition to produce two new functions, $f(g(x))$ and $g(f(x))$.
- To evaluate a composite function, $f(g(x))$, at a specific value, substitute the value into the equation for $g(x)$ and then substitute the result into $f(x)$ and evaluate, or determine the composite function first and then evaluate for the value of x.
- To determine the equation of a composite function, substitute the second function into the first as read from left to right. To compose $f(g(x))$, substitute the equation of $g(x)$ into the equation of $f(x)$.
- The domain of $f(g(x))$ is the set of all values of x in the domain of g for which $g(x)$ is in the domain of f. Restrictions on the inner function as well as the composite function must be considered.

Homework

#1-10 on page 507 (omit #6)

10.3 Composite Functions, pages 507 to 509

1. a) 3 **b)** 0 **c)** 2 **d)** -1

2. a) 2 **b)** 2 **c)** -4 **d)** -5

3. a) 10 **b)** -8 **c)** -2 **d)** 28

4. a) $f(g(a)) = 3a^2 + 1$ **b)** $g(f(a)) = 9a^2 + 24a + 15$

c) $f(g(x)) = 3x^2 + 1$ **d)** $g(f(x)) = 9x^2 + 24x + 15$

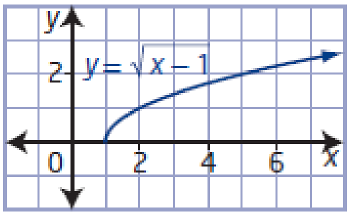
e) $f(f(x)) = 9x + 16$ **f)** $g(g(x)) = x^4 - 2x^2$

5. a) $f(g(x)) = x^4 + 2x^3 + 2x^2 + x$,
$g(f(x)) = x^4 + 2x^3 + 2x^2 + x$

b) $f(g(x)) = \sqrt{x^4 + 2}$, $g(f(x)) = x^2 + 2$

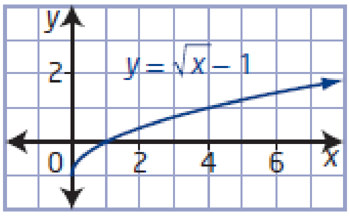
c) $f(g(x)) = x^2$, $g(f(x)) = x^2$

6. a)



domain
$\{x \mid x \geq 1, x \in R\}$,
range
$\{y \mid y \geq 0, y \in R\}$

b)



domain
$\{x \mid x \geq 0, x \in R\}$,
range
$\{y \mid y \geq -1, y \in R\}$

7. a) $g(x) = 2x - 5$ **b)** $g(x) = 5x + 1$

8. Christine is right. Ron forgot to replace all x's with the other function in the first step.

9. Yes. $k(j(x)) = j(k(x)) = x^6$; using the power law: $2(3) = 6$ and $3(2) = 6$.

10. No. $s(t(x)) = x^2 - 6x + 10$ and $t(s(x)) = x^2 - 2$.

11. a) $W(C(t)) = 3\sqrt{100 + 35t}$

b) domain $\{t \mid t \geq 0, t \in R\}$, range $\{W \mid W \geq 30, W \in W\}$