Combinations

Focus on...

- explaining the differences between a permutation and a combination
- determining the number of ways to select r elements from n different elements
- solving problems using the number of combinations of n different elements taken r at a time
- solving an equation that involves $_nC_r$ notation

Sometimes you must consider the order in which the elements of a set are arranged. In other situations, the order is not important. For example, when addressing an envelope, it is important to write the six-character postal code in the correct order. In contrast, addressing an envelope, affixing a stamp, and inserting the contents can be completed in any order.

In this section, you will learn about counting outcomes when order does not matter.

Investigate Making Selections When Order Is Not Important

Problem solving, reasoning, and decision-making are highly prized skills in today's workforce. Here is your opportunity to demonstrate those skills.

1. From a group of four students, three are to be elected to an executive committee with a specific position. The positions are as follows:

1st position	President
2nd position	Vice President
3rd position	Treasurer

   a) Does the order in which the students are elected matter? Why? yes

   b) In how many ways can the positions be filled from this group?

$_4P_3 = 24$ or $4! = 4 \times 3 \times 2 = 24$ permutions

2. Now suppose that the three students are to be selected to serve on a committee.

   a) Is the order in which the three students are selected still important? Why or why not? No

   b) How many committees from the group of four students are now possible? $_4C_3 = 4$ combinations

   c) How does your answer in part b) relate to the answer in step 1b)?

combination

- a selection of objects without regard to order
- all of the three-letter combinations of P, Q, R, and S are PQR, PQS, PRS, and QRS (arrangements such as PQR and RPQ are the same combination)

$n=4$
$r=3$

• PQR and RPQ represents 2 permutations but only 1 combination

Determining the Number of Possible Combinations

When counting with ***Permutations***, the order the objects are chosen is important. When the order of choosing does not have to be considered, we refer to ***Combinations***. A **combination** is a subset of the number of **permutations** and as such, the number of **combinations** for a particular situation is always less than the number of **permutations**.

The expression for evaluating **combinations** is as follows:

The notation ${}_nC_r$, or $\binom{n}{r}$, represents the number of combinations of n items taken r at a time, where $n \geq r$ and $r \geq 0$.

Why must $n \geq r \geq 0$?

$$
\begin{aligned}
{}_nC_r &= \frac{{}_nP_r}{r!} \\
&= \frac{\frac{n!}{(n-r)!}}{r!} \\
&= \frac{n!}{(n-r)!r!}
\end{aligned}
$$

$${}_nC_R = \frac{n!}{r!(n-r)!}$$

A combination is a selection of a group of objects, taken from a larger group, for which the kinds of objects selected is important, but not the order in which they are selected.

There are several ways to find the number of possible combinations. One is to use reasoning. Use the fundamental counting principle and divide by the number of ways that the objects can be arranged among themselves. For example, calculate the number of combinations of three digits made from the digits 1, 2, 3, 4, and 5 without repetitions:

Number of choices for the first digit	Number of choices for the second digit	Number of choices for the third digit
5	4	3

There are $5 \times 4 \times 3$ or 60 ways to arrange 3 items from 5. However, 3 digits can be arranged in 3! ways among themselves, and in a combination these are considered to be the same selection.

So,

$$\begin{aligned}\text{number of combinations} &= \frac{\text{number of permutations}}{3!} \\ &= \frac{60}{3!} \\ &= \frac{60}{6} \\ &= 10\end{aligned}$$

What does 3! represent?

- the number of ways the objects can be arranged among themselves

Example 1 order does not matter

A baseball team with 12 players is allowed to send four players to a weekend batting clinic. In how many ways can the group be chosen?

Solution $n = 12$ $r = 4$

Since order is not important, the group is a **combination**. You are choosing a **combination** of from a group of

$${}_nC_r = \frac{n!}{r!(n-r)!}$$

$${}_{12}C_4 = \frac{12!}{4!(12-4)!}$$

$${}_{12}C_4 = \frac{12!}{4!\,8!}$$

$${}_{12}C_4 = \mathbf{495}$$

Example 2 order does not matter

A committee of size 4 and a committee of size 3 are to be assigned from a group of 10 people. How many ways can this be done if no person is assigned to both committees?

Solution

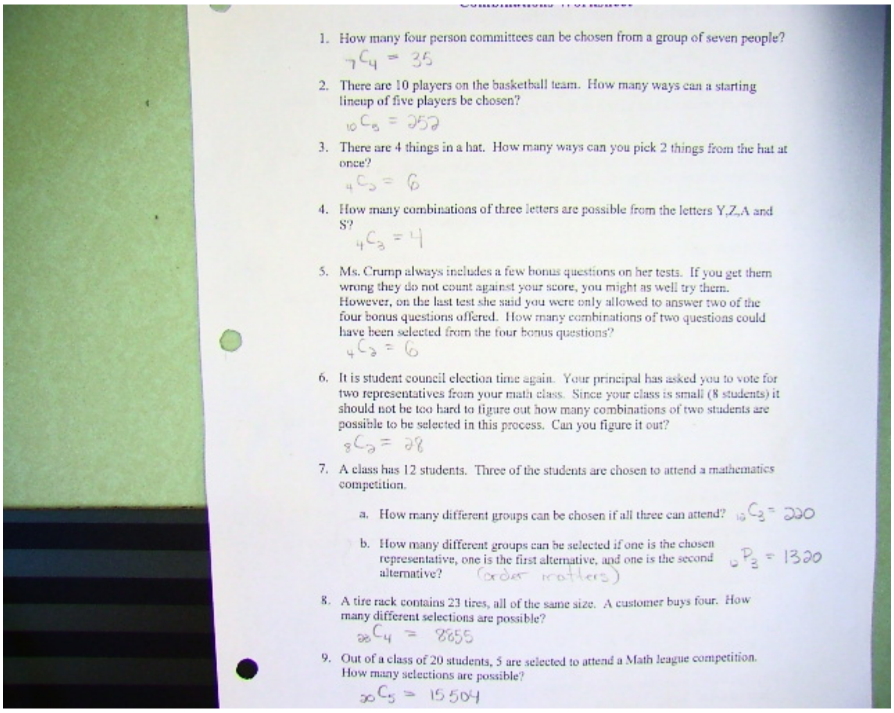
means multiply

1st Committee	**2nd Committee**	**Committee of size 4 AND Committee of size 3**
${}_nC_r = \dfrac{n!}{r!(n-r)!}$	${}_nC_r = \dfrac{n!}{r!(n-r)!}$	210×20
${}_{10}C_4 = \dfrac{10!}{4!(10-4)!}$	${}_6C_3 = \dfrac{6!}{3!(6-3)!}$	= 4200 ways
${}_{10}C_4 = \dfrac{10!}{4!\,6!}$	${}_6C_3 = \dfrac{6!}{3!\,3!}$	There are 4200 ways to form a committee of size 4 and a committee of size 3 from a group of 10 people if no person is assigned to both committees.
${}_{10}C_4 = 210$	${}_6C_3 = 20$	

* Note: "and" means multiply
"or" means add

Questions from Homework

Answers to Homework



1. How many four person committees can be chosen from a group of seven people?

   ${}_7C_4 = 35$

2. There are 10 players on the basketball team. How many ways can a starting lineup of five players be chosen?

   ${}_{10}C_5 = 252$

3. There are 4 things in a hat. How many ways can you pick 2 things from the hat at once?

   ${}_4C_2 = 6$

4. How many combinations of three letters are possible from the letters Y,Z,A and S?

   ${}_4C_3 = 4$

5. Ms. Crump always includes a few bonus questions on her tests. If you get them wrong they do not count against your score, you might as well try them. However, on the last test she said you were only allowed to answer two of the four bonus questions offered. How many combinations of two questions could have been selected from the four bonus questions?

   ${}_4C_2 = 6$

6. It is student council election time again. Your principal has asked you to vote for two representatives from your math class. Since your class is small (8 students) it should not be too hard to figure out how many combinations of two students are possible to be selected in this process. Can you figure it out?

   ${}_8C_2 = 28$

7. A class has 12 students. Three of the students are chosen to attend a mathematics competition.

   a. How many different groups can be chosen if all three can attend? ${}_{12}C_3 = 220$

   b. How many different groups can be selected if one is the chosen representative, one is the first alternative, and one is the second alternative? (order matters) ${}_{12}P_3 = 1320$

8. A tire rack contains 23 tires, all of the same size. A customer buys four. How many different selections are possible?

   ${}_{23}C_4 = 8855$

9. Out of a class of 20 students, 5 are selected to attend a Math league competition. How many selections are possible?

   ${}_{20}C_5 = 15504$

Answers to Homework

1. A hockey team has 17 players. Six of the players are selected, at random, to attend a summer hockey school. In how many different ways can the players be selected? $_{17}C_6 = 12\,376$

2. In the card game "120s," each player is dealt a hand of five cards. How many different hands can be given to the first player?
$_{52}C_6 = 2\,598\,960$

3. At a customer service counter, the customers usually take a numbered ticket from a machine so that they are served in order. On one day, however, the machine is broken. The clerk has eight customers.

Order matters

* (a) In how many different ways can the eight customers be served? $_8P_8 = 40\,320$
* (b) In how many ways can the clerk serve the first four customers? $_8P_4 = 1680$

4. A car dealer's lot contains 34 mid-sized cars. A rental agency purchases 12 of the vehicles. How many different possibilities are there for choosing the 12 vehicles? $_{34}C_{12} = 548\,354\,040$

5. Lotteries sometimes use "scratch and win" cards. These are cards in which a number of prize boxes are hidden and can be revealed by scratching away a waxy covering. If all of the revealed prize boxes match, you win a prize. Suppose that one such card contains eight prize boxes and you are to reveal any four. In how many ways can you do this? $_8C_4 = 70$

6. A chess club has six boys and six girls. Four players are selected to attend a chess clinic. If two boys and two girls are to attend, how many different selections are possible? $_6C_2 \times {}_6C_2 = 15 \times 15 = 225$

7. A hockey team has two goalies, six defense, and nine forwards. From the team, six people are chosen to attend a hockey school.

   (a) In how many ways can you select six players from the team, regardless of playing position? $_{17}C_6 = 12\,376$

   (b) In how many ways can you select the six players if exactly one goalie must attend? $_2C_1 \times {}_{15}C_5 = 2 \times 3003 = 6006$

8. Explain why a combination lock would be more correctly referred to as a "permutation" lock. order matters

Example 2

Combinations With Cases

Rianna is writing a geography exam. The instructions say that she must answer a specified number of questions from each section. How many different selections of questions are possible if

a) she must answer two of the four questions in part A and three of the five questions in part B?

b) she must answer two of the four questions in part A and at least four of the five questions in part B?

a) ${}_4C_2 \times {}_5C_3$

$= 6 \times 10$

$= 60$

b) Case 1: (4 of 5 from B)

$= {}_4C_2 \times {}_5C_4$

$= 6 \times 5$

$= 30$

Case 2: (5 of 5 from B)

$= {}_4C_2 \times {}_5C_5$

$= 6 \times 1$

$= 6$

$\boxed{30 + 6 = 36 \text{ combinations}}$

Example 3

Simplifying Expressions and Solving Equations With Combinations

a) Express as factorials and simplify $\dfrac{{}_nC_5}{{}_{n-1}C_3}$.

b) Solve for n if $2({}_nC_2) = {}_{n+1}C_3$.

a) $\dfrac{n!}{5!(n-5)!} \div \dfrac{(n-1)!}{3!(n-1-3)!}$

$= \dfrac{n!}{5!(n-5)!} \div \dfrac{(n-1)!}{3!(n-4)!}$

$= \dfrac{n!}{5!(n-5)!} \times \dfrac{3!(n-4)!}{(n-1)!}$

$= \dfrac{n\cancel{(n-1)!}}{5(4)\cancel{(3!)}\cancel{(n-5)!}} \times \dfrac{\cancel{3!}(n-4)\cancel{(n-5)!}}{\cancel{(n-1)!}}$

$= \boxed{\dfrac{n(n-4)}{20}}$

b) $2\left[\dfrac{n!}{2!(n-2)!}\right] = \dfrac{(n+1)!}{3!(n+1-3)!}$

$\cancel{2}\left[\dfrac{n!}{\cancel{2}(n-2)!}\right] = \dfrac{(n+1)!}{6(n-2)!}$

$\dfrac{n!}{\cancel{(n-2)!}} = \dfrac{(n+1)!}{6\cancel{(n-2)!}}$

$n! = \dfrac{(n+1)!}{6}$

$6n! = (n+1)!$

$6\cancel{n!} = (n+1)\cancel{n!}$

$6 = n+1$

$\boxed{5 = n}$

Homework

Page 534 #1, 4, 5, 6, 8, 11, 14, 15, 16, 17, 20

Answers to Homework

11.2 Combinations, pages 534 to 536

1. a) Combination, because the order that you shake hands is not important.

b) Permutation, because the order of digits is important.

c) Combination, since the order that the cars are purchased is not important.

d) Combination, because the order that players are selected to ride in the van is not important.

2. ${}_5P_3$ is a permutation representing the number of ways of arranging 3 objects taken from a group of 5 objects. ${}_5C_3$ is a combination representing the number of ways of choosing any 3 objects from a group of 5 objects. ${}_5P_3 = 60$ and ${}_5C_3 = 10$.

3. a) ${}_6P_4 = 360$ **b)** ${}_7C_3 = 35$

c) ${}_5C_2 = 10$ **d)** ${}_{10}C_7 = 120$

4. a) 210 **b)** 5040

5. a) AB, AC, AD, BC, BD, CD

b) AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC

c) The number of permutations is 2! times the number of combinations.

6. a) $n = 10$ **b)** $n = 7$ **c)** $n = 4$ **d)** $n = 5$

7. a) Case 1: one-digit numbers, Case 2: two-digit numbers, Case 3: three-digit numbers

b) Cases of grouping the 4 members of the 5-member team from either grade: Case 1: four grade 12s, Case 2: three grade 12s and one grade 11, Case 3: two grade 12s and two grade 11s, Case 4: one grade 12 and three grade 11s, Case 5: four grade 11s

8. Left Side $= {}_{11}C_3$ Right Side $= {}_{11}C_8$

$$= \frac{11!}{(11-3)!3!} \qquad = \frac{11!}{(11-8)!8!}$$

$$= \frac{11!}{8!3!} \qquad = \frac{11!}{3!8!}$$

Left Side = Right Side

9. a) ${}_5C_5 = 1$

b) ${}_5C_0 = 1$; there is only one way to choose 5 objects from a group of 5 objects and only one way to choose 0 objects from a group of 5 objects.

10. a) 4 **b)** 10

11. a) 15 **b)** 22

12. Left Side

$$= {}_nC_{r-1} + {}_nC_r$$

$$= \frac{n!}{(n-(r-1))!(r-1)!} + \frac{n!}{(n-r)!r!}$$

$$= \frac{n!}{(n-r+1)!(r-1)!} + \frac{n!}{(n-r)!r!}$$

$$= \frac{[n!(n-r)!r!] + [n!(n-r+1)!(r-1)!]}{(n-r+1)!(r-1)!(n-r)!r!}$$

$$= \frac{n!(n-r)!r(r-1)! + n!(n-r+1)(n-r)!(r-1)!}{(n-r+1)!(r-1)!(n-r)!r!}$$

$$= \frac{n!(n-r)!(r-1)![r + (n-r+1)]}{(n-r+1)!(r-1)!(n-r)!r!}$$

$$= \frac{n!\cancelto{1}{(n-r)!}\cancelto{1}{(r-1)!}(n+1)}{(n-r+1)\cancelto{1}{(r-1)!}\cancelto{1}{(n-r)!}r!}$$

$$= \frac{n!(n+1)}{(n-r+1)!r!}$$

$$= \frac{(n+1)!}{(n-r+1)!r!}$$

Right Side $= {}_{n+1}C_r$

$$= \frac{(n+1)!}{(n+1-r)!r!}$$

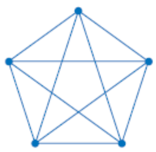
Left Side = Right Side

13. 20 different burgers; this is a combination because the order the ingredients is put on the burger is not important.

14. a) 210

b) combination, because the order of toppings on a pizza is not important

15. a) Method 1: Use a diagram.
Method 2: Use combinations.
${}_5C_2 = 10$, the same as the number of combinations of 5 people shaking hands.



b) 10

c) The number of triangles is given by ${}_{10}C_3 = \frac{10!}{(10-3)!3!} = \frac{10!}{7!3!}$. The number of lines is given by ${}_{10}C_2 = \frac{10!}{(10-2)!2!} = \frac{10!}{8!2!}$.
The number of triangles is determined by the number of selections with choosing 3 points from 10 non-collinear points, whereas the number of lines is determined by the number of selections with choosing 2 points from the 10 non-collinear points.

Answers to Homework

16. Left Side $= {}_nC_r$

$$= \frac{n!}{(n-r)!r!}$$

Right Side $= {}_nC_{n-r}$

$$= \frac{n!}{(n-(n-r))!(n-r)!}$$

$$= \frac{n!}{(n-n+r)!(n-r)!}$$

$$= \frac{n!}{r!(n-r)!}$$

Left Side = Right Side

17. a) 125 970 **b)** 44 352 **c)** 1945

18. a) 2 598 960 **b)** 211 926 **c)** 388 700

19. a) 525 **b)** 576

20. a) $\frac{40!}{20!20!}$ **b)** 116 280