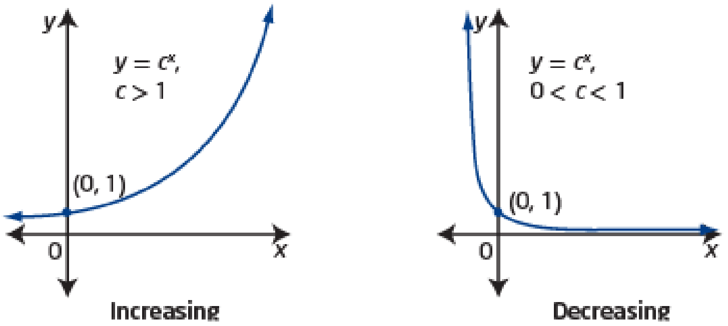
Exponential Functions

The graph of an **exponential function**, such as $y = c^x$, is increasing for $c > 1$, decreasing for $0 < c < 1$, and neither increasing nor decreasing for $c = 1$. From the graph, you can determine characteristics such as domain and range, any intercepts, and any asymptotes.

exponential function

- a function of the form $y = c^x$, where c is a constant ($c > 0$) and x is a variable



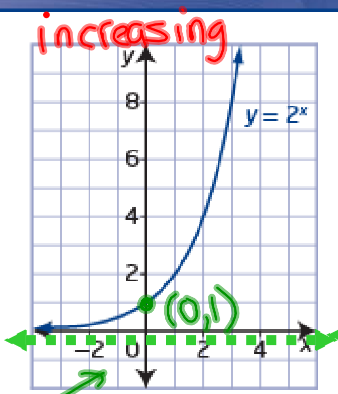
Why is the definition of an exponential function restricted to positive values of c?

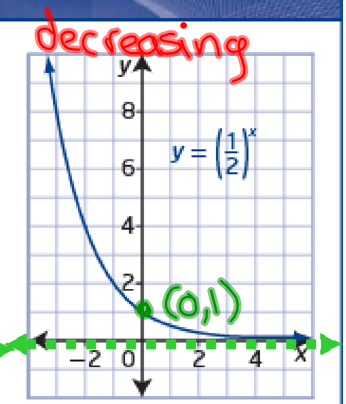
Did You Know?

Any letter can be used to represent the base in an exponential function. Some other common forms are $y = a^x$ and $y = b^x$. In this chapter, you will use the letter c. This is to avoid any confusion with the transformation parameters, a, b, h, and k, that you will apply in Section 7.2.

Key Ideas

- An exponential function of the form $y = c^x$, $c > 0$,
  - is increasing for $c > 1$
  - is decreasing for $0 < c < 1$
  - is neither increasing nor decreasing for $c = 1$
  - has a domain of $\{x \mid x \in \mathbb{R}\}$
  - has a range of $\{y \mid y > 0, y \in \mathbb{R}\}$
  - has a y-intercept of 1
  - has no x-intercept
  - has a horizontal asymptote at $y = 0$





Example 1

Analyse the Graph of an Exponential Function

Graph each exponential function. Then identify the following:

- the domain and range
- the x-intercept and y-intercept, if they exist
- whether the graph represents an increasing or a decreasing function
- the equation of the horizontal asymptote

a) $y = 4^x$

b) $f(x) = \left(\frac{1}{2}\right)^x$

a) $y=4^x$

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y>0, y \in R\}$

x-int: none

y-int: $y=1$ or $(0,1)$

increasing $(c=4)$

HA: $y=0$

b) $f(x)=\left(\frac{1}{2}\right)^x$

Domain: $\{x \mid x \in R\}$

Range: $\{y \mid y>0, y \in R\}$

x-int: none

y-int: $y=1$ or $(0,1)$

decreasing $(c=\frac{1}{2})$

HA: $y=0$

Solution

a) Method 1: Use Paper and Pencil

Use a table of values to graph the function.
Select integral values of x that make it easy to calculate the corresponding values of y for $y = 4^x$.

$y=4^x$

x	y
-2	$\frac{1}{16}$
-1	$\frac{1}{4}$
0	1
1	4
2	16

Domain: $\{x \mid x \in \mathbb{R}\}$

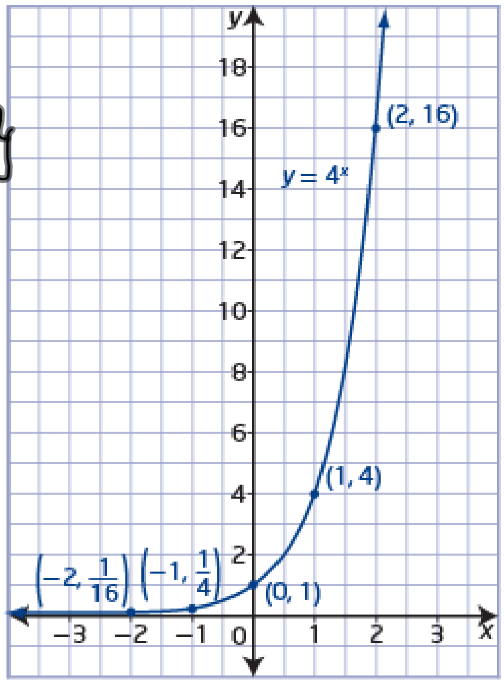
Range: $\{y \mid y>0, y \in \mathbb{R}\}$

x-int: none

y-int: $y=1$ or $(0,1)$

increasing $(c=4)$

HA: $y=0$

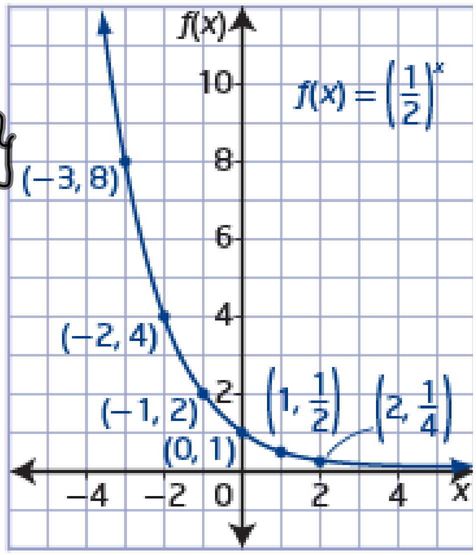


b) Method 1: Use Paper and Pencil
Use a table of values to graph the function.
Select integral values of x that make it easy to calculate the corresponding values of y for $f(x) = \left(\frac{1}{2}\right)^x$.

$y = \left(\frac{1}{2}\right)^x$

x	$f(x)$
−3	8
−2	4
−1	2
0	1
1	$\frac{1}{2}$
2	$\frac{1}{4}$

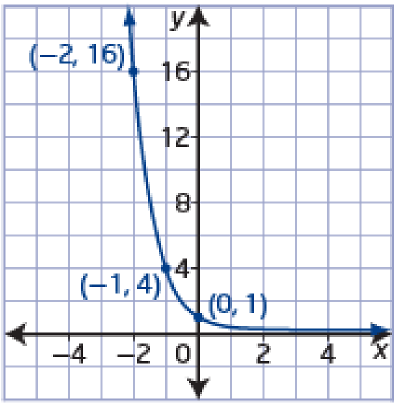
Domain: $\{x \mid x \in R\}$
Range: $\{y \mid y > 0, y \in R\}$
x-int: none
y-int: $y = 1$ or $(0, 1)$
decreasing $(c = \frac{1}{2})$
HA: $y = 0$



Example 2

Write the Exponential Function Given Its Graph

What function of the form $y = c^x$ can be used to describe the graph shown?



What is the value of c?

Solution

Look for a pattern in the ordered pairs from the graph.

x	*y*
−2	16
−1	4
0	1

$> \frac{1}{4}$
$> \frac{1}{4}$

16, 4, 1 → form a geometric sequence.

As the value of *x* increases by 1 unit, the value of *y* decreases by a factor of $\frac{1}{4}$. Therefore, for this function, $c = \frac{1}{4}$.

Choose a point other than (0, 1) to substitute into the function $y = \left(\frac{1}{4}\right)^x$ to verify that the function is correct. Try the point (−2, 16).

Why should you not use the point (0, 1) to verify that the function is correct?

(0,1) is common to all exponential functions of the form $y = c^x$, $c > 0$

Check:

Left Side   Right Side

$y = \left(\frac{1}{4}\right)^x$

Left Side	Right Side
16	$\left(\frac{1}{4}\right)^{-2}$
	$(4)^2$
	16 ✓

So, given that the original value is 1.5,

- if we know that the value **doubles** in **5** years, the equation is: $V = 1.5 \cdot 2^{\frac{x}{5}}$.
- if we know that the value **doubles** in **11** years, the equation is: $V = 1.5 \cdot 2^{\frac{x}{11}}$.
- if we know that the value **triples** in **7** years, the equation is: $V = 1.5 \cdot 3^{\frac{x}{7}}$.

Example 2

Initial Amount Base 2 exp. = $\frac{x}{7}$

Anita purchased a book for $13.50 in 1990. If the value of the book **doubled** every 7 years, how much would it be worth in 4 years, 11 years, 50 years?

exp

Solution: $V = (\text{Initial Amount})(\text{Base})^{\text{exp}}$

Since it states the value is doubled we can write the equation as: $V = 13.50 \cdot 2^{\frac{x}{7}}$.

So: after 4 years $V = 13.50 \cdot 2^{\frac{4}{7}} = \$20.06$

after 11 years $V = 13.50 \cdot 2^{\frac{11}{7}} = \$40.12$

after 50 years $V = 13.50 \cdot 2^{\frac{50}{7}} = \$1907.86$

Example 3

A culture is found to have 2300 bacteria. The number of bacteria **triples** in 4 h. Find the amount of bacteria at the end of one day.

Solution A = (Initial Amount)(Base)^{exp}

The equation for this will be: $A = 2300 \cdot 3^{\frac{x}{4}}$, where x is the # of hours. We use a base of 3 since we are given the tripling time.

So: In 24 hours: $A = 2300 \cdot 3^{\frac{24}{4}} = 1676700$ bacteria.

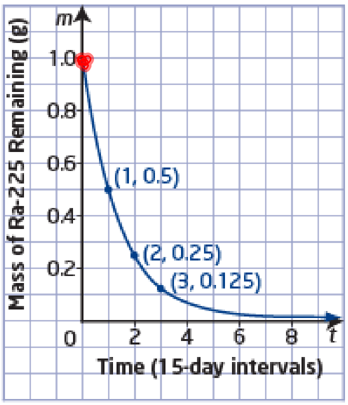
The three examples above are each exponential functions that exhibit **exponential growth.** We now look at some applications of exponential functions as they relate to **exponential decay.** (growth → increasing)

Base: $(c > 1)$

Example 3

Application of an Exponential Function

A radioactive sample of radium (Ra-225) has a half-life of 15 days. The mass, m, in grams, of Ra-225 remaining over time, t, in 15-day intervals, can be modelled using the exponential graph shown.



a) What is the initial mass of Ra-225 in the sample? What value does the mass of Ra-225 remaining approach as time passes? 1 gram

b) What are the domain and range of this function?

c) Write the exponential decay model that relates the mass of Ra-225 remaining to time.

d) Estimate how many days it would take for Ra-225 to decay to $\frac{1}{30}$ of its original mass.

b) D: $\{t \mid t \geq 0, t \in R\}$

R: $\{m \mid 0 < m \leq 1, m \in R\}$

c) Initial Amount = 1

Base = $\frac{1}{2} = 0.5$

exp = $\frac{x}{15}$

$y = 1\left(\frac{1}{2}\right)^{\frac{x}{15}}$ or $m = 1\left(\frac{1}{2}\right)^{\frac{t}{15}}$

d) $m = 1\left(\frac{1}{2}\right)^{\frac{t}{15}}$

$$\frac{\frac{1}{30}}{1} = \frac{\cancel{1}\left(\frac{1}{2}\right)^{t/15}}{\cancel{1}}$$

$$\frac{1}{30} = \left(\frac{1}{2}\right)^{t/15} \rightarrow \frac{\log(1/30)}{\log(1/2)} = 4.91$$

$$\left(\cancel{\tfrac{1}{2}}\right)^{4.91} = \left(\cancel{\tfrac{1}{2}}\right)^{t/15}$$

$$15 \cdot 4.91 = \frac{t}{\cancel{15}} \cdot \cancel{15}$$

$\boxed{73.6 = t}$ → 73.6 days for the Ra-225 to be $\frac{1}{30}$ of its initial mass.

Homework

7.1 Characteristics of Exponential Functions, pages 342 to 345

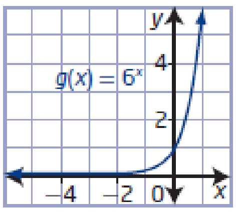
1. a) No, the variable is not the exponent.
b) Yes, the base is greater than 0 and the variable is the exponent.
c) No, the variable is not the exponent.
d) Yes, the base is greater than 0 and the variable is the exponent.

2. a) $f(x) = 4^x$ **b)** $g(x) = \left(\frac{1}{4}\right)^x$
c) $x = 0$, which is the y-intercept

3. a) B **b)** C **c)** A

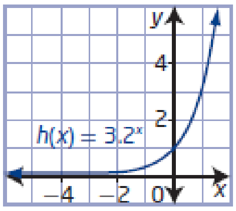
4. a) $f(x) = 3^x$ **b)** $f(x) = \left(\frac{1}{5}\right)^x$

5. a)



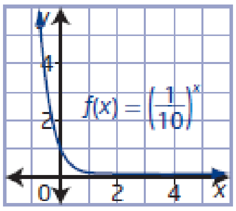
domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

b)



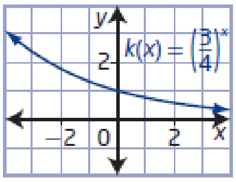
domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function increasing, horizontal asymptote $y = 0$

c)



domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

d)



domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y > 0, y \in \mathbb{R}\}$, y-intercept 1, function decreasing, horizontal asymptote $y = 0$

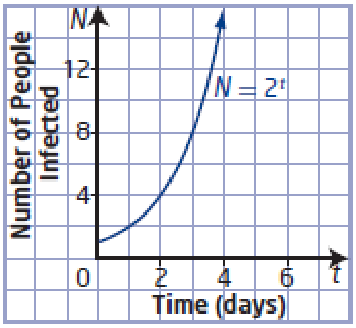
6. a) $c > 1$; number of bacteria increases over time

b) $0 < c < 1$; amount of actinium-225 decreases over time

c) $0 < c < 1$; amount of light decreases with depth

d) $c > 1$; number of insects increases over time

7. a)

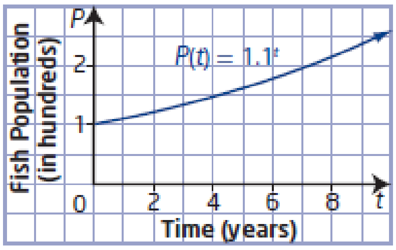


The function $N = 2^t$ is exponential since the base is greater than zero and the variable t is an exponent.

b) **i)** 1 person **ii)** 2 people **iii)** 16 people **iv)** 1024 people

8. a) If the population increases by 10% each year, the population becomes 110% of the previous year's population. So, the growth rate is 110% or 1.1 written as a decimal.

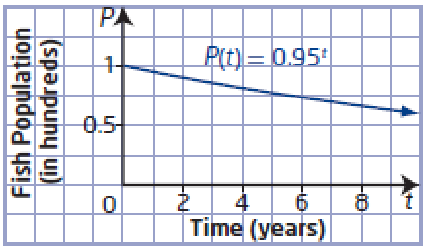
b)



domain $\{t \mid t \geq 0, t \in R\}$ and range $\{P \mid P \geq 100, P \in R\}$

c) The base of the exponent would become $100\% - 5\%$ or 95%, written as 0.95 in decimal form.

d)



domain $\{t \mid t \geq 0, t \in R\}$ and range $\{P \mid 0 < P \leq 100, P \in R\}$