10. The rate at which liquids cool can be modelled by an approximation of Newton's law of cooling, $T(t) = (T_i - T_f)(0.9)^{\frac{t}{5}} + T_f$, where T_f represents the final temperature, in degrees Celsius; T_i represents the initial temperature, in degrees Celsius; and t represents the elapsed time, in minutes. Suppose a cup of coffee is at an initial temperature of 95 °C and cools to a temperature of 20 °C.

a) State the parameters a, b, h, and k for this situation. Describe the transformation that corresponds to each parameter.

b) Sketch a graph showing the temperature of the coffee over a period of 200 min.

c) What is the approximate temperature of the coffee after 100 min?

d) What does the horizontal asymptote of the graph represent?

Given.

$T_i = 95°C$

$T_f = 20°C$

a) $T(t) = (T_i - T_f)(0.9)^{\frac{t}{5}} + T_f$

$T(t) = 75(0.9)^{\frac{t}{5}} + 20$

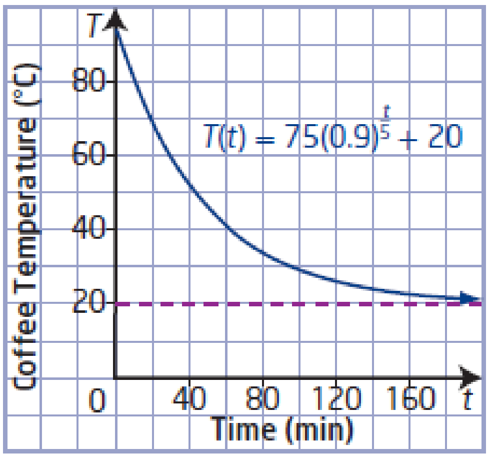
$a = 75 \rightarrow$ vertical stretch by a factor of 75

$b = \frac{1}{5} \rightarrow$ horizontal stretch by a factor of 5

$h = 0 \rightarrow$ no horizontal translation

$k = 20 \rightarrow$ translated 20 units up.

b)



c) $T(100) = 75(0.9)^{\frac{100}{5}} + 20$

$T(100) \doteq 29.12°C$

d) Final Temp.

Solving Exponential Equations

Focus on...

- determining the solution of an exponential equation in which the bases are powers of one another
- solving problems that involve exponential growth or decay
- solving problems that involve the application of exponential equations to loans, mortgages, and investments

Exponent Laws

① $x^2 \cdot x^3 = x^{2+3} = x^5$

② $\dfrac{x^2}{x^{1/3}} = x^{2-\frac{1}{3}} = x^{\frac{6}{3}-\frac{1}{3}} = x^{\frac{5}{3}}$

③ $(x^{-2})^5 = x^{-10} = \left(\dfrac{1}{x}\right)^{10} = \dfrac{1}{x^{10}}$

④ $\sqrt{x} = x^{\frac{1}{2}}$

$\sqrt[3]{x} = x^{1/3}$

$\sqrt[5]{x} = x^{1/5}$

$\sqrt[3]{x^2} = (x^2)^{1/3} = x^{2/3}$

$(\sqrt[5]{x})^3 = (x^{1/5})^3 = x^{3/5}$

Example 1

Change the Base of Powers

Rewrite each expression as a power with a base of 3.

a) 27

$\boxed{3^3}$

$\frac{\log 27}{\log 3} = 3$

b) 9^2

81

$\boxed{3^4}$

$(3^2)^2$

$\boxed{3^4}$

c) $27^{\frac{1}{3}}(\sqrt[3]{81})^2$

$3((81)^{1/3})^2$

$3(81)^{2/3}$

$3(3^4)^{2/3}$

$3^1(3)^{8/3}$

$3^{1+\frac{8}{3}}$

$3^{\frac{3}{3}+\frac{8}{3}}$

$\boxed{3^{\frac{11}{3}}}$

Example 2

Solve an Equation by Changing the Base

Solve each equation. (Solve for x) •Get a common base!

a) $4^{x+2} = 64^x$

$4^{x+2} = (4^3)^x$

$\cancel{4}^{x+2} = \cancel{4}^{3x}$

$x+2 = 3x$

$-2x = -2$

$x = 1$

$\boxed{\text{Test } x=1}$

LHS	RHS
4^{x+2}	64^x
4^{1+2}	64^1
4^3	64
64	

LHS = RHS

$x=1$ is a solution

b) $4^{2x} = 8^{2x-3}$

$(2^2)^{2x} = (2^3)^{2x-3}$

$\cancel{2}^{4x} = \cancel{2}^{6x-9}$

$4x = 6x - 9$

$-2x = -9$

$x = \frac{9}{2}$ or 4.5

$\boxed{\text{Test } x = \frac{9}{2}}$

LHS	RHS
4^{2x}	8^{2x-3}
$4^{2(\frac{9}{2})}$	$8^{2(\frac{9}{2})-3}$
4^9	8^{9-3}
262 144	8^6
	262 144

LHS = RHS

$x = \frac{9}{2}$ is a solution

Example 3

Solve Problems Involving Exponential Equations With Different Bases

Christina plans to buy a car. She has saved \$5000. The car she wants costs \$5900. How long will Christina have to invest her money in a term deposit that pays 6.12% per year, compounded quarterly, before she has enough to buy the car?

Solution

The formula for compound interest is $A = P(1 + i)^n$, where A is the amount of money at the end of the investment; P is the principal amount deposited; i is the interest rate per compounding period, expressed as a decimal; and n is the number of compounding periods. In this problem:

Divide the interest rate by 4 because interest is paid quarterly or four times a year.

Key Ideas

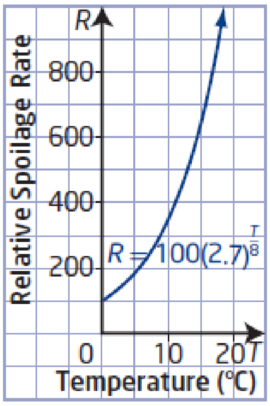
- Some exponential equations can be solved directly if the terms on either side of the equal sign have the same base or can be rewritten so that they have the same base.
  - If the bases are the same, then equate the exponents and solve for the variable.
  - If the bases are different but can be rewritten with the same base, use the exponent laws, and then equate the exponents and solve for the variable.
- Exponential equations that have terms with bases that you cannot rewrite using a common base can be solved approximately. You can use either of the following methods:
  - Use systematic trial. First substitute a reasonable estimate for the solution into the equation, evaluate the result, and adjust the next estimate according to whether the result is too high or too low. Repeat this process until the sides of the equation are approximately equal.
  - Graph the functions that correspond to the expressions on each side of the equal sign, and then identify the value of x at the point of intersection, or graph as a single function and find the x-intercept.

Homework

#1-10 on page 364 (omit #7)

7.3 Solving Exponential Equations, pages 364 to 365

1. a) 2^{12} b) 2^9 c) 2^{-6} d) 2^4
2. a) 2^3 and 2^4 b) 3^{2x} and 3^3
   c) $\left(\frac{1}{2}\right)^{2x}$ and $\left(\frac{1}{2}\right)^{2x-2}$ d) 2^{-3x+6} and 2^{4x}
3. a) 4^2 b) $4^{\frac{2}{3}}$ c) 4^3 d) 4^3
4. a) $x = 3$ b) $x = -2$ c) $w = 3$ d) $m = \frac{7}{4}$
5. a) $x = -3$ b) $x = -4$ c) $y = \frac{11}{4}$ d) $k = 9$
6. a) 10.2 b) 11.5 c) −2.8 d) 18.9
7. a) 58.71 b) −1.66 c) −5.38 d) −8
   e) 2.71 f) 14.43 g) −3.24 h) −1.88
8. a)



   b) approximately 5.6 °C
   c) approximately 643
   d) approximately 13.0 °C
9. 3 h
10. 4 years
11. a) $A = 1000(1.02)^n$ b) \$1372.79 c) 9 years
12. a) $C = \left(\frac{1}{2}\right)^{\frac{t}{5.3}}$ b) $\frac{1}{32}$ of the original amount
    c) 47.7 years
13. a) $A = 500(1.033)^n$ b) \$691.79
    c) approximately 17 years
14. \$5796.65