5.4 Optimization Problems I: Creating the Model

GOAL

Create models to represent optimization problems.

Let x = # of bracken fern
Let y = # of baby's breath

EXPLORE...

- A florist is ordering bracken fern and baby's breath for bouquets and centrepieces.
- No more than 100 stems of baby's breath will be ordered.
- More than 100 stems of bracken fern will be ordered.
- The florist has space to store no more than 250 stems, in total.
- Is each of the following a combination she can order? Explain.

$y \leq 100$
$x > 100$
$x + y \leq 250$

Baby's Breath (y)	Bracken Fern (x)				
0	150	✓			
25	25	✗			
50	150	✓			
100	100	✗			
95.5	114.5	✗			
100	150	✓			
150	125	✗			

SAMPLE ANSWER

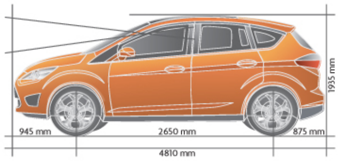
Only points (0, 150), (50, 150), (100, 150) are possible solutions because they are in the solution region and belong to the correct number set (whole numbers). Points (25, 25), (100, 100), and (150, 125) are not in the solution region, even though they have whole-number coordinates. Point (95.5, 114.5) looks like it is a possible solution. However, it is not from the correct number set, so it is not a solution.

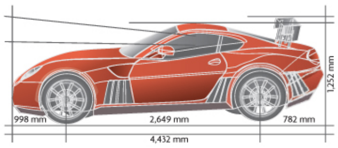
INVESTIGATE the Math

A toy company manufactures two types of toy vehicles: sport-utility vehicles x and racing cars. y

- Because the supply of materials is limited, no more than 40 racing cars y and 60 sport-utility vehicles x can be made each day.
- However, the company can make 70 or more vehicles, in total, each day.
- It costs \$8 to make a racing car and \$12 to make a sport-utility vehicle. ✱

There are many possible combinations of racing cars and sport-utility vehicles that could be made. The company wants to know what combinations will result in the minimum and maximum costs, and what those costs will be.





? How can this situation be modelled?

A. What are the two variables in this situation?

B. Write a system of three linear inequalities to represent these conditions:
- the total number of racing cars that can be made $y \le 40$
- the total number of sport-utility vehicles that can be made $x \le 60$
- the total number of vehicles that can be made $x + y \ge 70$

C. What do you know about the restrictions on the domain and range of the variables? Explain.

D. Graph the system. Choose at least two points in the solution region that are possible solutions to the system.

E. What quantity in this situation needs to be minimized and maximized? Write an equation to represent how the two variables relate to this quantity.

Answers

A. number of sports utility vehicles and number of racing cars

B. Let x represent the number of sports utility vehicles, and let y represent the number of racing cars.

Total number of racing cars: $y \le 40$ — solid

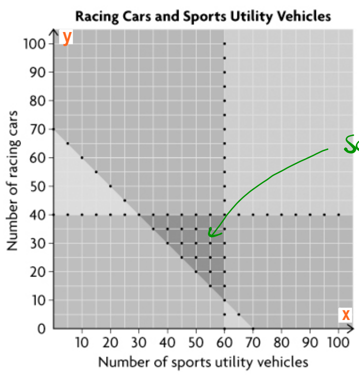
Total number of sports utility vehicles: $x \le 60$ — solid

Total number of vehicles: $y + x \ge 70$ — solid

C. The restrictions are $x \in W$ and $y \in W$, because only whole numbers of vehicles make sense.

D. e.g., (50, 20) and (40, 35)

$y \le 40$
$x \le 60$
$y + x \ge 70$



solution set

(stippled because $x \in W$, $y \in W$)

E. The total cost of producing the two types of toys, C, must be optimized:

$C = 8y + 12x$ (Objective Function)

Do not graph

Reflecting

F. Each combination below is a possible solution to the system of linear inequalities:

i) 40 racing cars and 60 sport-utility vehicles
ii) 40 racing cars and 30 sport-utility vehicles
iii) 10 racing cars and 60 sport-utility vehicles
iv) 30 racing cars and 40 sport-utility vehicles

Use your equation from part E to calculate the manufacturing cost for each solution. What do you notice?

Answers

F. i) \$1040 $\rightarrow$ $8y + 12x = 8(40) + 12(60) = 320 + 720 = 1040$
ii) \$680
iii) \$800
iv) \$720

Each combination costs a different amount; the combination of 40 racing cars and 30 sports utility vehicles costs the least of these combinations, and 40 racing cars and 60 sport-utility vehicles costs the most. The other combinations are in between.

APPLY the Math

EXAMPLE 1 Creating a model for an optimization problem with whole-number variables

Three teams are travelling to a basketball tournament in minivans x and cars. y

- Each team has no more than 2 coaches and 14 athletes.
- Each car can take 4 team members, and each minivan can take 6 team members.
- No more than 4 minivans and 12 cars are available.

The school wants to know the combination of cars and minivans that will require the minimum and maximum number of vehicles. Create a model to represent this situation.

Juanita's Solution

Let x represent the number of minivans.
Let y represent the number of cars.

The two variables in the problem are the number of cars and the number of minivans. The values of these variables are whole numbers.

$x \in W$ and $y \in W$

Constraints:
Number of cars available:
$y \leq 12$
Number of minivans available:
$x \leq 4$
Number of team members:
$4y + 6x \leq 48$

I knew that this is an **optimization problem** because the number of vehicles has to be minimized and maximized.

I wrote three linear inequalities to represent the three limiting conditions, or **constraints**.

The maximum number of team members is the number of teams multiplied by the maximum number of coaches and athletes:
$3(14) + 3(2) = 48$

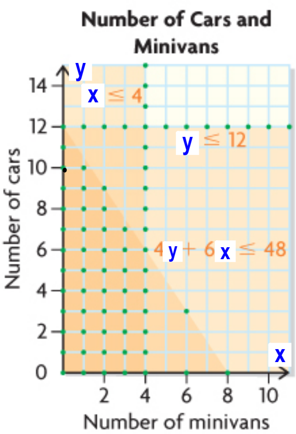
optimization problem
A problem where a quantity must be maximized or minimized following a set of guidelines or conditions.

constraint
A limiting condition of the optimization problem being modelled, represented by a linear inequality.

Objective function:
Let V represent the total number of vehicles.
$V = y + x$

I created an equation, called the **objective function**, to represent the relationship between the two variables (number of minivans and number of cars) and the quantity to be minimized and maximized (number of vehicles).

objective function
In an optimization problem, the equation that represents the relationship between the two variables in the system of linear inequalities and the quantity to be optimized.



I graphed the system of three inequalities.

One of the solutions in the **feasible region** represents the combination of cars and minivans that results in the minimum total number of vehicles and another solution represents the maximum. I think I could use the objective function to determine each point, but I am not certain how yet.

feasible region
The solution region for a system of linear inequalities that is modelling an optimization problem.

Let x represent the number of minivans.
Let y represent the number of cars.

$x \in W$ and $y \in W$

Constraints:
Number of cars available:
$y \leq 12$
Number of minivans available:
$x \leq 4$
Number of team members:
$4y + 6x \leq 48$

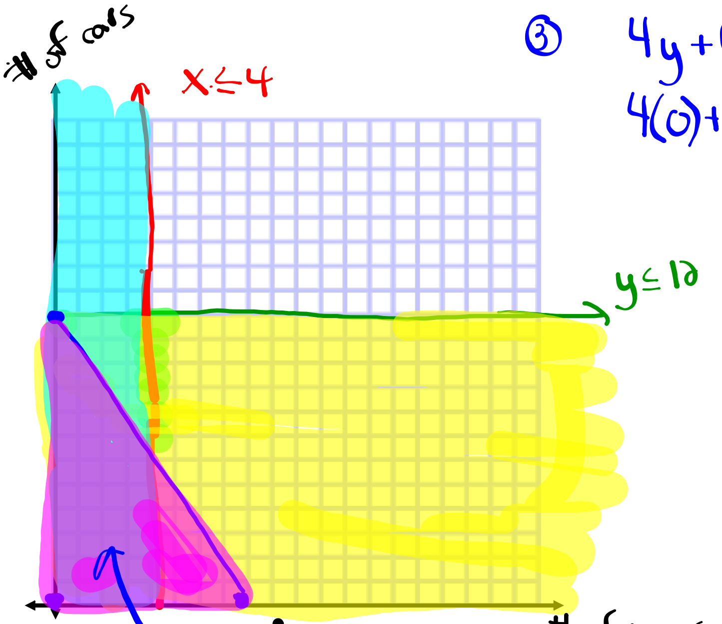
$y \leq 12$
① $y = 12$

$x \leq 4$
① $x = 4$

$4y + 6x \leq 48$
① $4y + 6x = 48$

② x int (y=0)
$4(0) + 6x = 48$
$\frac{6x}{6} = \frac{48}{6}$
$x = 8$
(8,0)

y int (x=0)
$4y + 6(0) = 48$
$\frac{4y}{4} = \frac{48}{4}$
$y = 12$
(0,12)



③ $4y + 6x \leq 48$

$4(0) + 6(0)$	48
0	

True

Since 0<48 shade the plane where (0,0) lies

solution set is stippled because x∈W, y∈W

In Summary

Key Ideas

- To solve an optimization problem, you need to determine which combination of values of two variables results in a maximum or minimum value of a related quantity.
- When creating a model, the first step is to represent the situation algebraically. An algebraic model includes these parts:
  - a defining statement of the variables used in your model
  - a statement describing the restrictions on the variables
  - a system of linear inequalities that describes the constraints
  - an objective function that shows how the variables are related to the quantity to be optimized
- The second step is to represent the system of linear inequalities graphically.
- In optimization problems, any restrictions on the variables are considered constraints. For example, if you are working with positive real numbers, $x \geq 0$ and $y \geq 0$ are constraints and should be included in the system of linear inequalities.

Need to Know

- You can create a model for an optimization problem by following these steps:

  Step 1. Identify the quantity that must be optimized. Look for key words, such as *maximize* or *minimize*, *largest* or *smallest*, and *greatest* or *least*.

  Step 2. Define the variables that affect the quantity to be optimized. Identify any restrictions on these variables.

  Step 3. Write a system of linear inequalities to describe all the constraints of the problem. Graph the system.

  Step 4. Write an objective function to represent the relationship between the variables and the quantity to be optimized.

Assignment: **page 248**

Questions 1, 2, and 3

$x =$ apples

$y =$ oranges

b) (i) $x \geq 5$

(ii) $y \geq 6$

(iii) $0.20x + 0.35y \leq 7$

$20x + 35y \leq 700$

SOLUTIONS => 5.4 Optimization Problems I: Creating the Model

1. Baskets of fruit are being prepared to sell.

- Each basket contains at least 5 apples and at least 6 oranges.
- Apples cost 20¢ each, and oranges 35¢ each. The budget allows no more than \$7, in total, for the fruit in each basket.

Answer each part below to create a model that could be used to determine the combination of apples and oranges that will result in the maximum number of pieces of fruit in a basket.

a) What are the two variables in this situation? Describe any restrictions.

Defining Statements

Let x represent the number of apples.
Let y represent the number of oranges.

Restrictions $x \in W, y \in W$ (# of apples/oranges must be positive)

b) Write a system of linear inequalities to represent each constraint:

Constraints

i) the number of apples in each basket.
$\{(x, y) \mid x \geq 5, x \in W, y \in W\}$
Solid Line

ii) the number of oranges in each basket.
$\{(x, y) \mid y \geq 6, x \in W, y \in W\}$
Solid Line

iii) the cost of each basket (in cents).

$\{(x, y) \mid 20x + 35y \leq 700, x \in W, y \in W\}$

Solid line

d) Graph the system.

① Equations of boundaries:

↳ $x = 5$ ↳ $y = 6$ ↳ $20x + 35y = 700$

② Two points on each boundary (x-int & y-int):

↳ $x = 5$ (O.K) * Vertical line

↳ $y = 6$ (O.K) * Horizontal Line.

↳ $20x + 35y = 700$

x-int:

$\frac{\cancel{20}x}{\cancel{20}} = \frac{700}{20}$

$x = 35$

y-int:

$\frac{\cancel{35}y}{\cancel{35}} = \frac{700}{35}$

$y = 20$

③ Test Point (0,0):

↳ $x \geq 5$
* Test point not required (Shade on the right of the boundary)

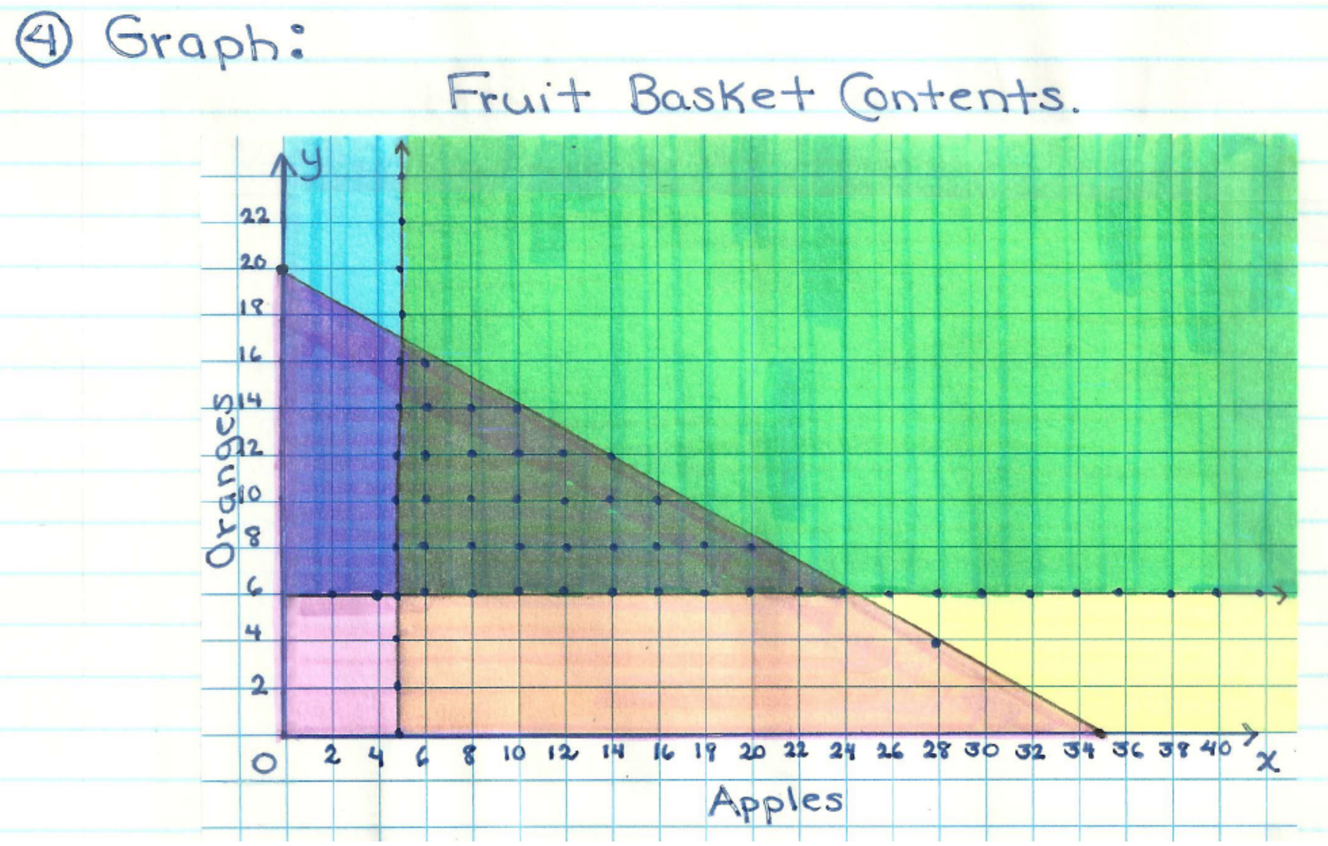
↳ $y \geq 6$
* Test point not required (Shade above the boundary)

↳ $20x + 35y \leq 700$

L.S	R.S.
$20x+35y$	700
$20(0)+35(0)$	
$0+0$	
0	

Since $0 \leq 700$, (0,0) is located in the solution region.

④ Graph:


Fruit Basket Contents.
y
Oranges
22
20
18
16
14
12
10
8
6
4
2
0
2 4 6 8 10 12 14 16 18 20 22 24 26 28 30 32 34 36 38 40
x
Apples

d) Write the objective function that represents how the quantity to be maximized relates to the variables.

Let N represent the number of pieces of fruit in the basket.

Objective function: $N = x + y$

2. A fast-food concession stand sells hotdogs and hamburgers.

- Daily sales can be as high as 300 hamburgers and hot dogs combined.
- The stand has room to stock no more than 200 hot dogs and no more than 150 hamburgers.
- Hot dogs are sold for \$3.25, and hamburgers are sold for \$4.75.

Create a model that could be used to determine the combination of hamburgers and hot dogs that will result in maximum sales.

SOLUTION

Defining Statements

Let x represent the number of hotdogs.
Let y represent the number of hamburgers.

Let R represent the sales revenue.

Constraints Restrictions

$\{(x,y) \mid x \le 200,\ x \in W,\ y \in W\}$

$\{(x,y) \mid y \le 150,\ x \in W,\ y \in W\}$

$\{(x,y) \mid x + y \le 300,\ x \in W,\ y \in W\}$

Objective Function: $R = 3.25x + 4.75y$

To Graph:

① Equations of boundaries:

↳ $x = 200$ ↳ $y = 150$ ↳ $x + y = 300$

② Two points on each boundary (x-int & y-int):

↳ $x = 200$ (O.K.)
* Vertical Line

↳ $y = 150$ (O.K.)
* Horizontal Line

↳ $x + y = 300$

x-int:
$x + 0 = 300$
$x = 300$

y-int:
$0 + y = 300$
$y = 300$

③ Test Point (0,0):

↳ $x \le 200$
* Test point not required. (shade on the left of the boundary).

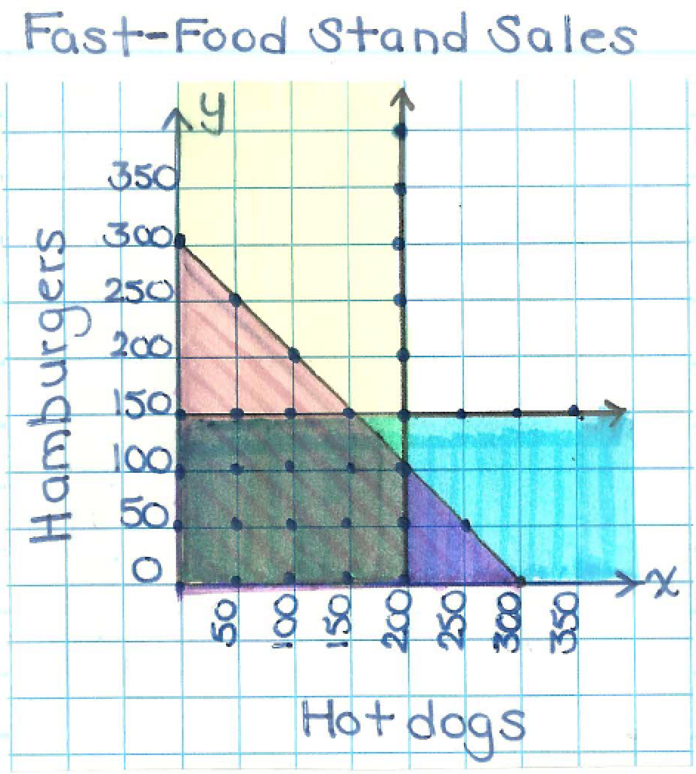
↳ $y \le 150$
* Test point not required. (shade below the boundary)

↳ $x + y \le 300$

L.S.	R.S.
$x+y$	300
$0+0$	
0	

Since $0 \le 300$, (0,0) is located in the solution region.

④ Graph:


Fast-Food Stand Sales
y
350
300
250
200
150
100
50
0
Hamburgers
x
50
100
150
200
250
300
350
Hot dogs

3. A vending machine sells juice and pop.

- The machine holds, at most, 240 cans of drinks.
- Sales from the vending machine show that at least 2 cans of juice are sold for each can pop.
- Each can of juice sells for \$1.00, and each can of pop sells for \$1.25.

Create a model that could be used to determine the maximum revenue from the vending machine.

Solution

Defining Statements

Let x represent the number of cans of juice.
Let y represent the number of cans of pop

Let R represent the revenue.

Constraints **Restrictions**

$\{(x,y) \mid x \geq 2y,\ x \in W,\ y \in W\}$

$\{(x,y) \mid x + y \leq 240,\ x \in W,\ y \in W\}$

Objective Function: $R = 1.00x + 1.25y$

To Graph:

① Equations of boundaries:

↳ $x = 2y$ ↳ $x + y = 240$

② Two points on each boundary (x-int & y-int):

↳ $x=2y$

x-int:
$x = 2(0)$
$x = 0$

y-int:
$\frac{0}{2} = \frac{\cancel{2}y}{\cancel{2}}$
$0 = y$

* We need to find another point.

If $x = 20$;
$\frac{20}{2} = \frac{\cancel{2}y}{\cancel{2}}$
$10 = y$

(20, 10)

↳ $x+y=240$

x-int:
$x + 0 = 240$
$x = 240$

y-int:
$0 + y = 240$
$y = 240$

3 Test Point $(0,20)$:

L.S.	R.S.
x	$2y$
0	$2(20)$
	40

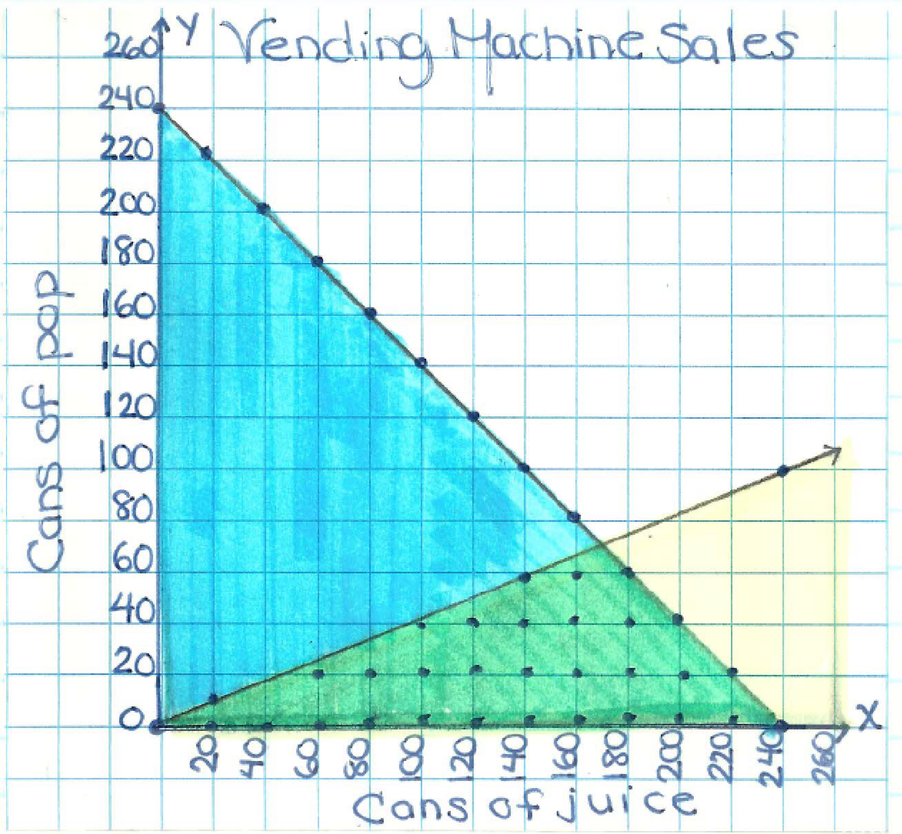
Since $0 \le 40$, $(0,20)$ is not located in the solution region.

Test Point $(0,0)$:

L.S.	R.S.
$x+y$	240
$0+0$	
0	

Since $0 \le 240$, $(0,0)$ is located in the solution region.

4 Graph:


Vending Machine Sales
Cans of pop
Cans of juice
Y
X
0
20
40
60
80
100
120
140
160
180
200
220
240
260

Attachments

6Ws4e1.mp4

6Ws4e2.mp4