Warm Up

2. Factor each of the following:

$x^{27}-1 \longrightarrow (a^3-b^3)=(a-b)(a^2+ab+b^2)$

$\underline{(x^9-1)}(x^{18}+x^9+1)$

$\underline{(x^3-1)}(x^6+x^3+1)(x^{18}+x^9+1)$

$\boxed{(x-1)(x^2+x+1)(x^6+x^3+1)(x^{18}+x^9+1)}$

$(x^2+1)^{\frac{1}{2}}+3(x^2+1)^{-\frac{1}{2}}$ Common Factor of $(x^2+1)^{-1/2}$

$(x^2+1)^{-1/2}\left[(x^2+1)^1+3(x^2+1)^0\right]$

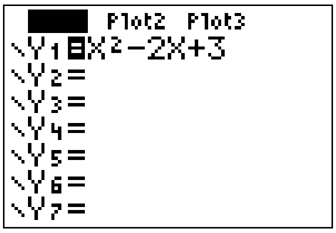
$(x^2+1)^{-1/2}(x^2+1+3)$

$(x^2+1)^{-1/2}(x^2+4)$

$\frac{x^2+4}{(x^2+1)^{1/2}}$ or $\frac{x^2+4}{\sqrt{x^2+1}}$

Limit of a Function

Let's examine the function $f(x) = x^2 - 2x + 3$



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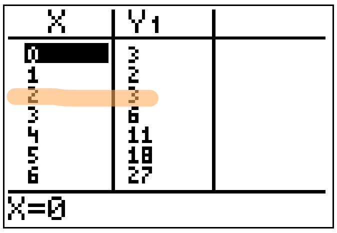
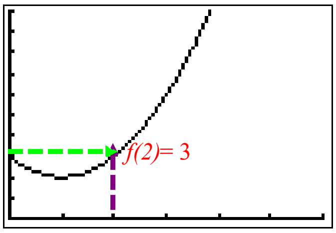
Plot1 Plot2 Plot3
\Y1=X²-2X+3
\Y2=
\Y3=
\Y4=
\Y5=
\Y6=
\Y7=
```

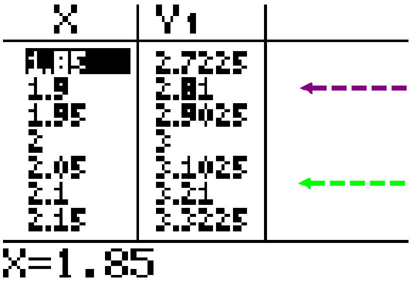


X	Y1
0	3
1	2
2	3
3	6
4	11
5	18
6	27

X=0



We can see that *f(2)*= 3...let's check the behaviour of *f* as we get closer and closer to *x* = 2.



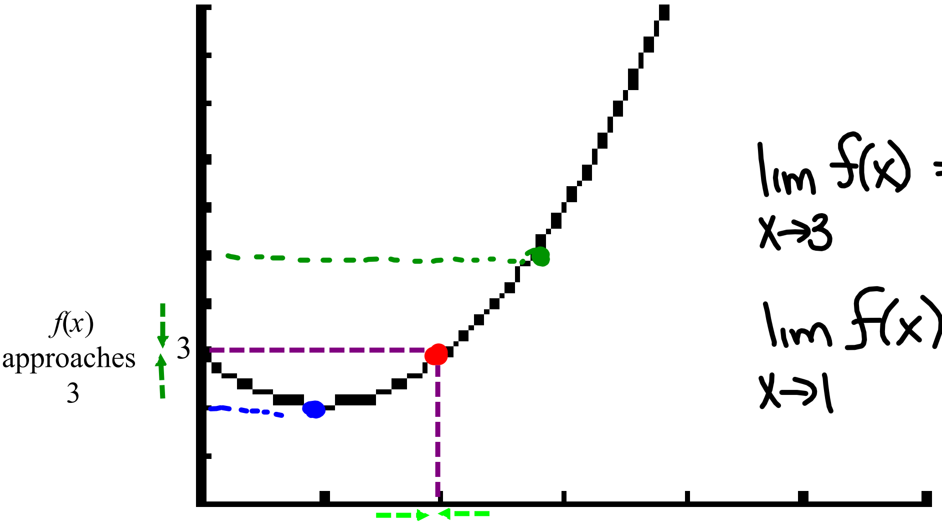
X	Y1
1.85	2.7225
1.9	2.81
1.95	2.9025
2	3
2.05	3.1025
2.1	3.21
2.15	3.3225

X=1.85

As *x* gets closer to 2 from the left *y* is getting closer to 3.

As *x* gets closer to 2 from the right *y* is getting closer to 3.

From the above, the notion of the limit of a function arises...



As *x* approaches 2

$$\lim_{x\to 3} f(x) = 5$$

$$\lim_{x\to 1} f(x) = 2$$

Notation: $\lim_{x\to 2} f(x) = 3$

"The limit of the function *f*(*x*) as *x* approaches 2 is equal to 3."

The common sense definition of a limit...

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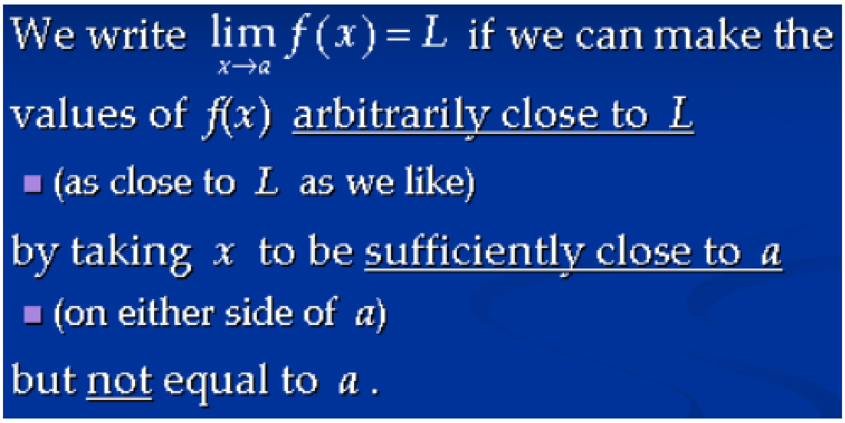
What *is* a limit?

A formal definition of a limit...



We write $\lim_{x\to a} f(x) = L$ if we can make the values of $f(x)$ arbitrarily close to L

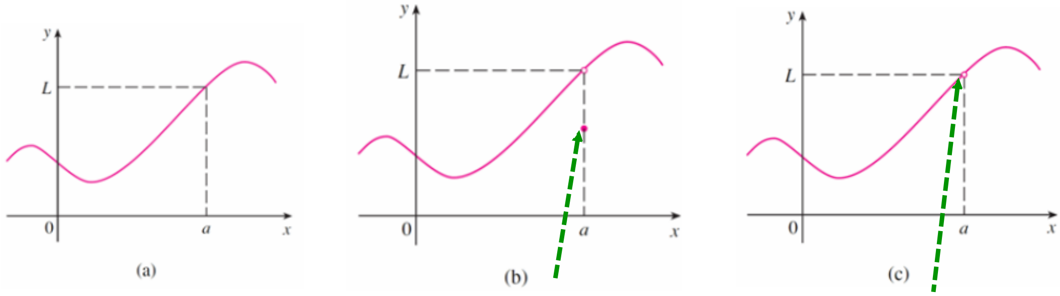
- (as close to L as we like)

by taking x to be sufficiently close to a

- (on either side of a)

but not equal to a.

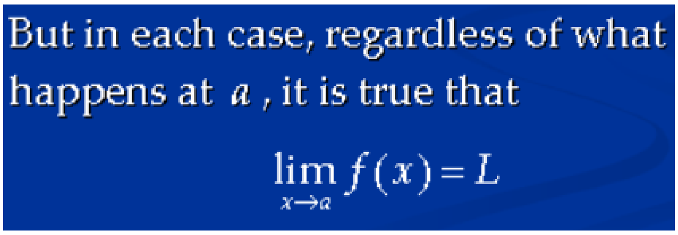
Look at the graphs of these three functions...



(a) (b) (c)

Notice $f(a) \neq L$

Notice $f(a)$ is undefined



But in each case, regardless of what happens at a, it is true that

$$\lim_{x\to a} f(x) = L$$

Evaluating Limits

I. Using a Graph:

- We looked at this in the previous two examples

II. Algebraically:

- Direct Substitution...

Examples:

$$\lim_{x\to -2}\frac{x^2-2x+1}{x+3}$$

$$\lim_{x\to -2}\frac{(-2)^2-2(-2)+1}{(-2)+3}=\frac{4+4+1}{1}=\boxed{9}$$

$$\lim_{x\to 3}\left(16-x^2\right)$$

$$\lim_{x\to 3}\left(16-(3)^2\right)=16-9=\boxed{7}$$

- Indeterminate limits... $\Rightarrow$Direct substitution leads to $\frac{0}{0}$

⇒ Factor
⇒ Rationalize
⇒ Expand
⇒ Find Common Denominators

Examples:

$$\lim_{x\to 4}\frac{x^2-16}{x-4}$$

$$\lim_{x\to 4}\frac{(x+4)\cancel{(x-4)}}{\cancel{(x-4)}}$$

$$\lim_{x\to 4} x+4 = 4+4=\boxed{8}$$

$$\lim_{h\to 0}\frac{\left(\sqrt{4+h}-2\right)}{h}\frac{\left(\sqrt{4+h}+2\right)}{\left(\sqrt{4+h}+2\right)}$$

$$\lim_{h\to 0}\frac{4+h-4}{h\left(\sqrt{4+h}+2\right)}$$

$$\lim_{h\to 0}\frac{\cancel{h}}{\cancel{h}\left(\sqrt{4+h}+2\right)}=\boxed{\frac{1}{4}}$$

Try these...remember to use your algebra skills
to try and eliminate the**indeterminate form**.

$$\lim_{x\to 2}\frac{(x+2)^2-16}{x^2-4}$$

Expand

Factor

$$\lim_{x\to 2}\frac{x^2+4x+4-16}{(x-2)(x+2)}$$

$$\lim_{x\to 2}\frac{x^2+4x-12}{(x-2)(x+2)}$$

Factor

$$\lim_{x\to 2}\frac{(x+6)\cancel{(x-2)}}{\cancel{(x-2)}(x+2)}=\frac{8}{4}=\boxed{2}$$

$$\lim_{x\to -2}\frac{x^4-16}{x^3+8}$$

Factor

$$\lim_{x\to -2}\frac{(x^2-4)(x^2+4)}{(x+2)(x^2-2x+4)}$$

$$\lim_{x\to -2}\frac{\cancel{(x+2)}(x-2)(x^2+4)}{\cancel{(x+2)}(x^2-2x+4)}$$

$$=\frac{(-4)(8)}{12}=-\frac{32}{12}=\boxed{\frac{-8}{3}}$$

$$\lim_{x\to 0}\frac{x^2+3x}{(x+2)^2-(x-2)^2}$$

Common Factor

Expand

$$\lim_{x\to 0}\frac{x(x+3)}{x^2+4x+4-(x^2-4x+4)}$$

$$\lim_{x\to 0}\frac{x(x+3)}{x^2+4x+4-x^2+4x-4}$$

$$\lim_{x\to 0}\frac{\cancel{x}(x+3)}{8\cancel{x}}$$

$$\lim_{x\to 0}\frac{x+3}{8}=\boxed{\frac{3}{8}}$$

$$2x\cdot\frac{1}{x}-\frac{1}{2}\cdot 2x$$ CD: 2x

$$\lim_{x\to 2}\frac{\frac{1}{x}-\frac{1}{2}}{(x-2)\,2x}$$

$$\lim_{x\to 2}\frac{\cancel{(2-x)}^{-1}}{2x\cancel{(x-2)}^{1}}$$

$$\lim_{x\to 2}\frac{-1}{2x}=\boxed{\frac{-1}{4}}$$

Homework

④g) $\lim_{x\to 9} \frac{(x-9)(\sqrt{x}+3)}{(\sqrt{x}-3)(\sqrt{x}+3)}$

$\lim_{x\to \underline{9}} \frac{\cancel{(x-9)}(\sqrt{\underline{x}}+3)}{\cancel{(x-9)}} = 6$

$\lim_{x\to 9} \frac{x-9}{\sqrt{x}-3}$

$\lim_{x\to 9} \frac{(\sqrt{x}+3)\cancel{(\sqrt{x}-3)}}{\cancel{(\sqrt{x}-3)}} = 6$

④ b) $\lim_{x\to 1} \frac{x^2-3x+2}{x-1}$

$\lim_{x\to 1} \frac{\cancel{(x-1)}(x-2)}{\cancel{(x-1)}} = -1$

⑤ e) $\lim_{h\to 0} \frac{(\sqrt{9+h}-3)}{h}\frac{(\sqrt{9+h}+3)}{(\sqrt{9+h}+3)}$

$\lim_{h\to 0} \frac{9+h-9}{h(\sqrt{9+h}+3)}$

$\lim_{h\to \underline{0}} \frac{\cancel{h}}{\cancel{h}(\sqrt{9+\underline{h}}+3)} = \frac{1}{6}$

⑤ f) $\lim_{h\to 0} \frac{4(2+h)^2\cdot\frac{1}{(2+h)^2} - \frac{1}{4}\cdot 4(2+h)^2}{h\cdot 4(2+h)^2}$

$\lim_{h\to 0} \frac{4-(2+h)^2}{4h(2+h)^2}$ ← Diff of Squares

$\lim_{h\to 0} \frac{(2+(2+h))(2-(2+h))}{4h(2+h)^2}$

$\lim_{h\to 0} \frac{(4+\underline{h})\overset{-1}{\cancel{(-h)}}}{4\cancel{h}(2+\underline{h})^2} = \frac{-4}{16} = \frac{-1}{4}$

⑥ c) $\lim_{x\to 1} \frac{x^4-1}{x-1}$

$\lim_{x\to 1} \frac{(x^2-1)(x^2+1)}{(x-1)}$

$\lim_{x\to \underline{1}} \frac{\cancel{(x-1)}(x+1)(x^2+1)}{\cancel{(x-1)}} = 4$

⑥ e) $\lim_{x\to 1} \frac{x^2+x-2}{x^2-2x+1}$

$\lim_{x\to \underline{1}} \frac{(\underline{x}+2)\cancel{(x-1)}}{(\underline{x}-1)\cancel{(x-1)}} = \frac{3}{0} = \text{DNE}$