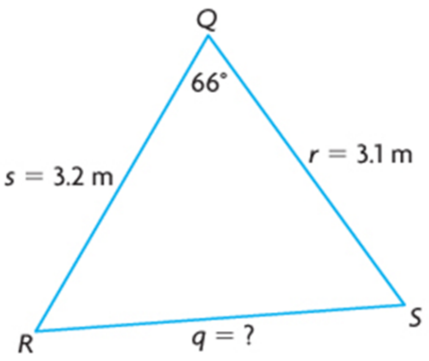
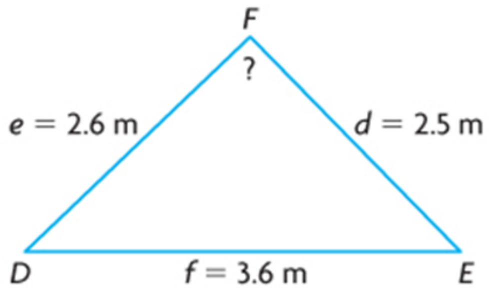
3.3 Proving and Applying the Cosine Law

INVESTIGATE the Math

The sine law cannot always help you determine unknown angle measures or side lengths. Consider these triangles:

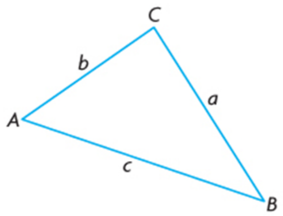


where $\dfrac{3.1}{\sin R} = \dfrac{3.2}{\sin S} = \dfrac{q}{\sin 66°}$



where $\dfrac{\sin E}{2.6} = \dfrac{\sin D}{2.5} = \dfrac{\sin F}{3.6}$

There are two unknowns in each pair of equivalent ratios, so the pairs cannot be used to solve for the unknowns. Another relationship is needed. This relationship is called the **cosine law**, and it is derived from the Pythagorean theorem.



LAW OF COSINES

Missing Side

$$a^2 = b^2 + c^2 - 2bc \cos A$$

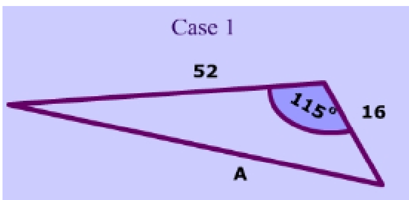
$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

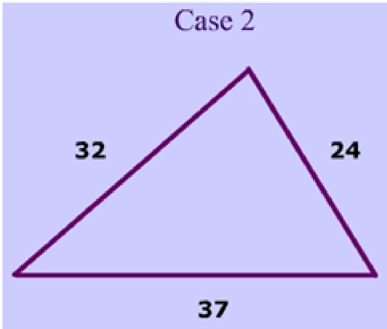
When will you use the Law of Cosines?

You will use the Law of Cosines when:

A) you need to find a missing side and you are given the other two sides and the angle in between them (the included angle).



B) you need to find an angle measure when all three side lengths are given.



For case (B) the formula can be rearranged in the following forms:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

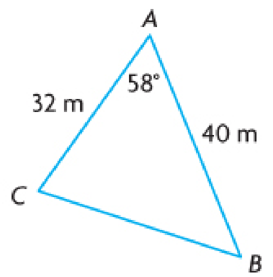
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

You may need to combine both the Law of Sines and the Law of Cosines in the same question☺

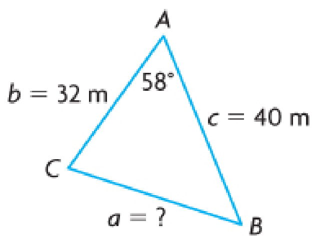
APPLY the Math

EXAMPLE 1 Using reasoning to determine the length of a side

Determine the length of CB to the nearest metre.



Justin's Solution



I labelled the sides with letters.
I couldn't use the sine law, because I didn't know a side length and the measure of its opposite angle.

$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$a^2 = 32^2 + 40^2 - 2(32)(40) \cos 58°$$

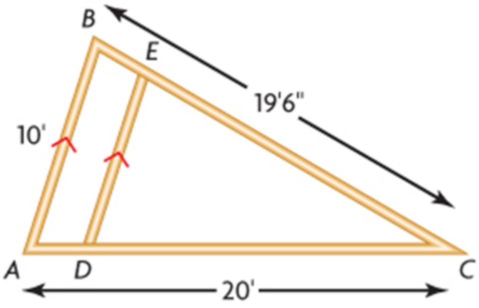
I knew the lengths of two sides (b and c) and the measure of the contained angle between these sides ($\angle A$). I had to determine side a, which is opposite $\angle A$. I chose the form of the cosine law that includes these four values. Then I substituted the values I knew into the cosine law.

$$a^2 = 1024 + 1600 - 2560 \cos 58°$$
$$a^2 = 2624 - 2560 \cos 58°$$
$$a^2 = 1267.406...$$
$$a = \sqrt{1267.406...}$$
$$a = 35.600...$$

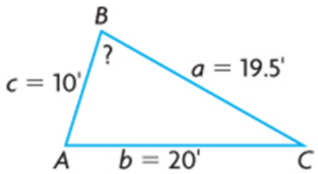
CB is 36 m.

EXAMPLE 2 Using reasoning to determine the measure of an angle

The diagram at the right shows the plan for a roof, with support beam *DE* parallel to *AB*. The local building code requires the angle formed at the peak of a roof to fall within a range of 70° to 80° so that snow and ice will not build up. Will this plan pass the local building code?



Emilie's Solution:



I drew a diagram, labelling the sides and angles. I wrote all the side lengths in terms of feet.

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{19.5^2 + 10^2 - 20^2}{2(19.5)(10)}$$

$$\cos B = \frac{80.25}{390}$$

$$\cos B = 0.2057...$$

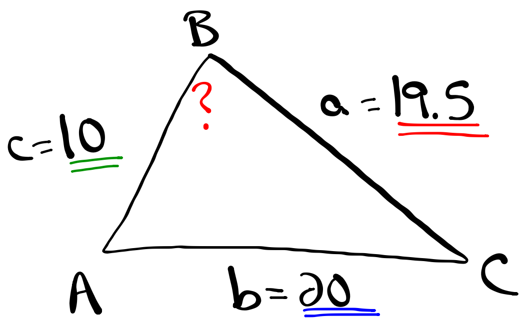
Since I wanted to determine $\angle B$ and I knew the length of all three sides, I wrote the form of the cosine law that contains $\angle B$.

$$\angle B = \cos^{-1}(0.2057...)$$

$$\angle B = 78.125...°$$

The angle formed at the peak of the roof is 78°. This plan passes the local building code.

I rounded to the nearest degree. The value of this angle is within the acceptable range.



$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos B = \frac{(19.5)^2 + (10)^2 - (20)^2}{2(19.5)(10)}$$

$$\cos B = \frac{380.25 + 100 - 400}{390}$$

$$\cos B = \frac{80.25}{390}$$

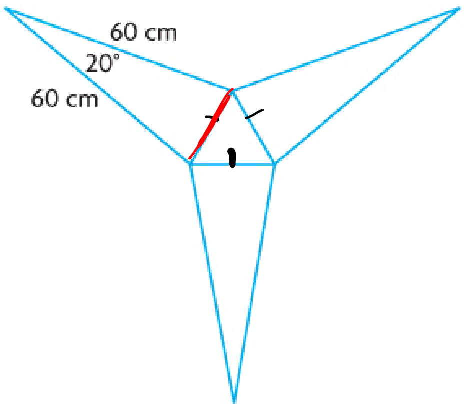
$$\cos B = 0.2058$$

$$\angle B = \cos^{-1}(0.2058)$$

$$\angle B = 78^\circ$$

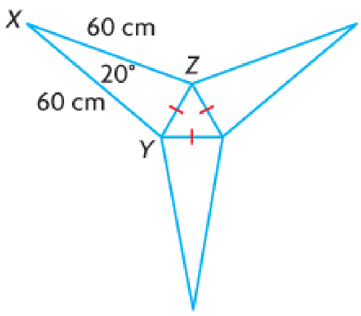
EXAMPLE 3 Solving a problem using the cosine law

A three-pointed star is made up of an equilateral triangle and three congruent isosceles triangles. Determine the length of each side of the equilateral triangle in this three-pointed star. Round the length to the nearest centimetre.



The world's largest Ukrainian Easter egg (called a pysanka) is located in Vegreville, Alberta. It is decorated with 2208 equilateral triangles and

Dakoda's Solution



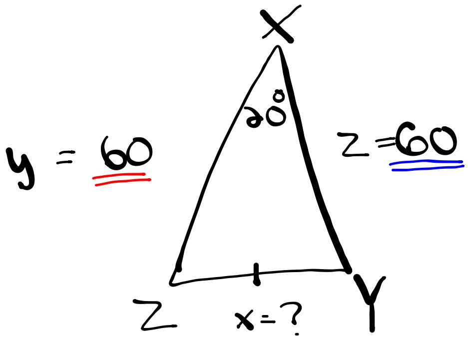
I named the vertices of one of the isosceles triangles.

$(x)^2 = (y)^2 + (z)^2 - 2(y)(z)\cos X$
$(x)^2 = 60^2 + 60^2 - 2(60)(60)\cos 20°$
$(x)^2 = 3600 + 3600 - 6765.786...$

I knew two sides and the contained angle in each isosceles triangle, so I used the cosine law to write an equation that involved *YZ*. Then I substituted the information that I knew.

$(x)^2 = 434.213...$
$x = \sqrt{434.213...}$
$x = 20.837...$

Each side of the equilateral triangle has a length of 21 cm.



$$x^2 = y^2 + z^2 - 2yz\cos X$$

$$x^2 = (60)^2 + (60)^2 - 2(60)(60)\cos 20^\circ$$

$$x^2 = 3600 + 3600 - 6765.78$$

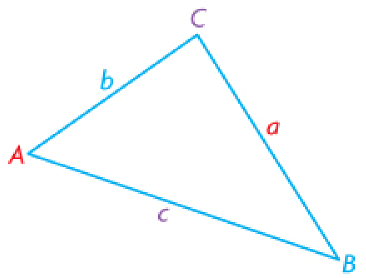
$$\sqrt{x^2} = \sqrt{434.22}$$

$$\boxed{x = 20.8}$$

In Summary

Key Idea

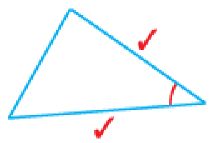
- The cosine law can be used to determine an unknown side length or angle measure in an acute triangle.

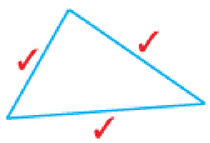


$$a^2 = b^2 + c^2 - 2bc \cos A$$
$$b^2 = a^2 + c^2 - 2ac \cos B$$
$$c^2 = a^2 + b^2 - 2ab \cos C$$

Need to Know

- You can use the cosine law to solve a problem that can be modelled by an acute triangle when you know:
  - two sides and the contained angle.
  - all three sides.



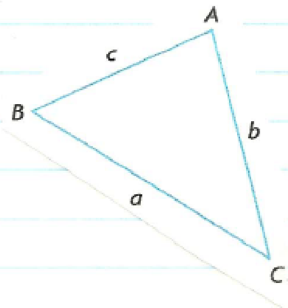


- The contained angle is the angle between two known sides.
- When using the cosine law to determine an angle, you can:
  - substitute the known values first, then solve for the unknown angle.
  - rearrange the formula to solve for the cosine of the unknown angle, then substitute and evaluate.

<u>Assignment:</u> **pgs. 136 - 138**
1, 2, 3, 4, 5, 7, 8

SOLUTIONS => 3.3 Proving and Applying the Cosine Law

1. Suppose that you are given each set of data for $\triangle ABC$. Can you use the cosine law to determine c? Explain.



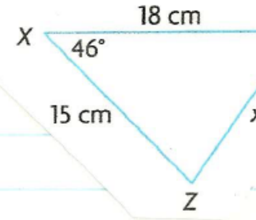
a) $a = 5\text{cm}, \angle A = 52°, \angle C = 43°$

No, you cannot use the cosine law to determine c. You do not have two sides and the contained angle.

b) $a = 7\text{cm}, b = 5\text{cm}, \angle C = 43°$

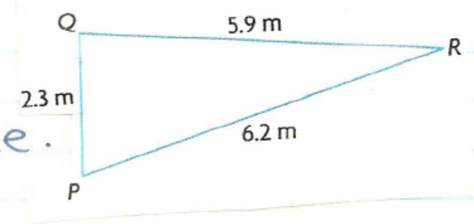
Yes, you can use the cosine law to determine c since you are given two sides and the contained angle.

2. Determine the length of side x to the nearest centimeter.



$$x^2 = y^2 + z^2 - 2yz\cos X$$
$$x^2 = (15)^2 + (18)^2 - 2(15)(18)\cos 46°$$
$$x^2 = 225 + 324 - 540(0.6947)$$
$$x^2 = 549 - 375.138$$
$$x^2 = 173.862$$
$$x = \sqrt{173.862}$$
$$x = 13.2 \text{ cm}$$
$$x = 13 \text{ cm}$$

3. Determine the measure of <P to the nearest degree.



$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos P = \frac{(6.2)^2 + (2.3)^2 - (5.9)^2}{2(6.2)(2.3)}$$

$$\cos P = \frac{38.44 + 5.29 - 34.81}{28.52}$$

$$\cos P = \frac{8.92}{28.52}$$

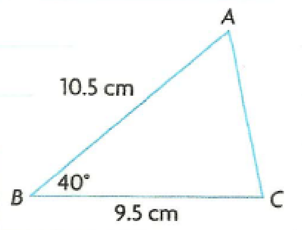
$$\cos P = 0.3128$$

$$\angle P = \cos^{-1}(0.3128)$$

$$\angle P = 72^\circ$$

4. Determine each unknown side length to the nearest tenth of a centimeter.

a)



$b^2 = a^2 + c^2 - 2ac\cos B$

$b^2 = (9.5)^2 + (10.5)^2 - 2(9.5)(10.5)\cos 40°$

$b^2 = 90.25 + 110.25 - 199.5(0.7660)$

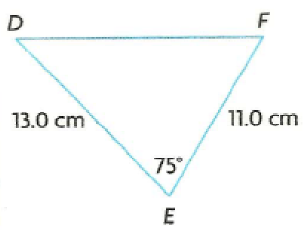
$b^2 = 200.5 - 152.817$

$b^2 = 47.683$

$b = \sqrt{47.683}$

$b = 6.9$ cm

b)



$e^2 = d^2 + f^2 - 2df\cos E$

$e^2 = (11.0)^2 + (13.0)^2 - 2(11.0)(13.0)\cos 75°$

$e^2 = 121 + 169 - 286(0.2588)$

$e^2 = 290 - 74.0168$

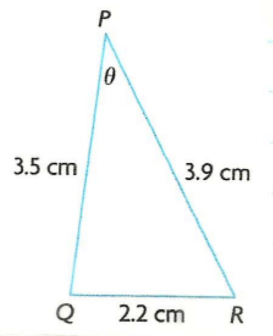
$e^2 = 215.9832$

$e = \sqrt{215.9832}$

$e = 14.7$ cm

5. Determine the measure of each indicated angle to the nearest degree.

a)



$$\cos P = \frac{q^2 + r^2 - p^2}{2qr}$$

$$\cos P = \frac{(3.9)^2 + (3.5)^2 - (2.2)^2}{2(3.9)(3.5)}$$

$$\cos P = \frac{15.21 + 12.25 - 4.84}{27.3}$$

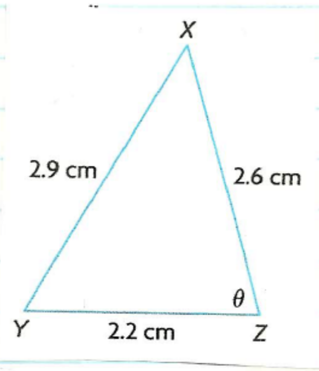
$$\cos P = \frac{22.62}{27.3}$$

$$\cos P = 0.8286$$

$$P = \cos^{-1}(0.8286)$$

$$P = 34°$$

b)



$$\cos Z = \frac{x^2 + y^2 - z^2}{2xy}$$

$$\cos Z = \frac{(2.2)^2 + (2.6)^2 - (2.9)^2}{2(2.2)(2.6)}$$

$$\cos Z = \frac{4.84 + 6.76 - 8.41}{11.44}$$

$$\cos Z = \frac{3.19}{11.44}$$

$$\cos Z = 0.2788$$

$$Z = \cos^{-1}(0.2788)$$

$$Z = 74°$$

7. Solve each triangle. Round all answers to the nearest tenth of a unit. (Round Angles to the nearest degree)

a) In $\triangle DEF$, $d = 5.0$cm, $e = 6.5$cm, and $\angle F = 65°$

$f^2 = d^2 + e^2 - 2de\cos F$
$f^2 = (5.0)^2 + (6.5)^2 - 2(5.0)(6.5)\cos 65°$
$f^2 = 25 + 42.25 - 65(0.4226)$
$f^2 = 67.25 - 27.469$
$f^2 = 39.781$
$f = \sqrt{39.781}$
$f = 6.3$ cm

$\cos E = \frac{d^2 + f^2 - e^2}{2df}$
$\cos E = \frac{(5.0)^2 + (6.3)^2 - (6.5)^2}{2(5.0)(6.3)}$
$\cos E = \frac{25 + 39.69 - 42.25}{63}$
$\cos E = \frac{22.44}{63}$
$\cos E = 0.3562$
$E = \cos^{-1}(0.3562)$
$E = 69°$

$\angle D = 180° - 65° - 69°$
$\angle D = 46°$

c) In $\triangle LMN$, $l = 5.5$ cm, $m = 4.6$ cm, and $n = 3.3$ cm

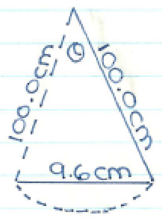
$$\cos L = \frac{m^2 + n^2 - l^2}{2mn}$$
$$\cos L = \frac{(4.6)^2 + (3.3)^2 - (5.5)^2}{2(4.6)(3.3)}$$
$$\cos L = \frac{21.16 + 10.89 - 30.25}{30.36}$$
$$\cos L = \frac{1.8}{30.36}$$
$$\cos L = 0.0593$$
$$L = \cos^{-1}(0.0593)$$
$$L = 87°$$

$$\cos M = \frac{n^2 + l^2 - m^2}{2nl}$$
$$\cos M = \frac{(3.3)^2 + (5.5)^2 - (4.6)^2}{2(3.3)(5.5)}$$
$$\cos M = \frac{10.89 + 30.25 - 21.16}{36.3}$$
$$\cos M = \frac{19.98}{36.3}$$
$$\cos M = 0.5504$$
$$M = \cos^{-1}(0.5504)$$
$$M = 57°$$

$N = 180° - 87° - 57°$
$N = 36°$

8. The pendulum of a grandfather clock is 100.0 cm long. When the pendulum swings from one side to the other side, the horizontal distance it travels is 9.6 cm.

a) Draw a diagram of the situation.



b) Determine the angle through which the pendulum swings. Round your answer to the nearest tenth of a degree.

$$\cos\theta = \frac{(100.0)^2 + (100.0)^2 - (9.6)^2}{2(100.0)(100.0)}$$

$$\cos\theta = \frac{10\,000 + 10\,000 - 92.16}{20\,000}$$

$$\cos\theta = \frac{19\,907.84}{20\,000}$$

$$\cos\theta = 0.9954$$

$$\theta = \cos^{-1}(0.9954)$$

$$\theta = 5.5^\circ \text{ or } 6^\circ$$

PM11-3s3.gsp

3s3e1 final.mp4

3s3e2 final.mp4