4.2 Proving and Applying the Sine and Cosine Laws for Obtuse Triangles

GOAL

Explain steps in the proof of the sine and cosine laws for obtuse triangles, and apply these laws to situations that involve obtuse triangles.

APPLY the Math

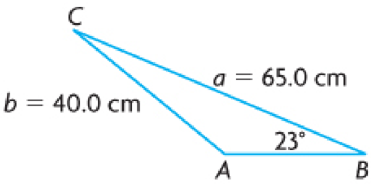
EXAMPLE 1 Use reasoning and the sine law to determine the measure of an obtuse angle

Look for this

In an obtuse triangle, $\angle B$ measures 23.0° and its opposite side, b, has a length of 40.0 cm. Side a is the longest side of the triangle, with a length of 65.0 cm. Determine the measure of $\angle A$ to the nearest tenth of a degree.

(Angle A is largest)

Bijan's Solution



I drew an obtuse triangle to represent $\triangle ABC$.

I knew that the longest side is always opposite the largest angle, so the 65.0 cm side must be opposite the obtuse angle, $\angle A$.

Since $\triangle ABC$ is not a right triangle, I knew that I could not use the primary trigonometric ratios to determine the measure of $\angle A$.

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

I noticed that the diagram has two side-angle pairs with only one unknown, $\angle A$. I decided to use the sine law.

$$\frac{\sin A}{65.0} = \frac{\sin 23°}{40.0}$$

The measure of an angle is the unknown, so I used the form of the sine law that has the angles in the numerator.

$$\frac{\cancel{40.0} \sin A}{\cancel{40.0}} = \frac{65.0 \sin 23}{40.0}$$

$$\sin A = 0.6349\ldots$$

I isolated $\sin A$.

$$\angle A = \sin^{-1}(0.6349\ldots)$$
$$\angle A = 39.4153\ldots°$$

I used the inverse sine to determine the measure of $\angle A$.

$$\angle A = 180° - 39.4153\ldots°$$
$$\angle A = 140.5846\ldots°$$

My reasoning suggests that $\angle A$ must be the obtuse angle. I used the relationship $\sin A = \sin(180° - A)$.

$\angle A$ measures 140.6°.

The measure of the angle seems appropriate, according to my diagram.

Given:

$b = 40\text{cm}$

$\angle B = 23^\circ$

$a = 65\text{cm}$

$\boxed{\angle A = 140.6^\circ}$

$\angle C = 180^\circ - 140.6 - 23^\circ$
$= 15.4^\circ$

$$\frac{\sin A}{a} = \frac{\sin B}{b}$$

$$\frac{\sin A}{65} = \frac{\sin 23^\circ}{40}$$

$$\frac{\cancel{40}\sin A}{\cancel{40}} = \frac{65\sin 23^\circ}{40}$$

$$\sin A = 0.6349$$

$$A = \sin^{-1}(0.6349)$$

$$A = 39.41^\circ$$

Because a is the largest side, $\angle A$ is the largest angle (obtuse)

$\angle A = 180^\circ - 39.4^\circ = \boxed{141.6^\circ}$

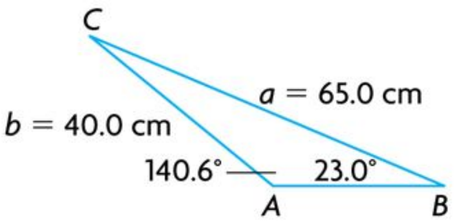
EXAMPLE 1 Use reasoning and the sine law to determine the measure of an obtuse angle

In an obtuse triangle, $\angle B$ measures 23.0° and its opposite side, b, has a length of 40.0 cm. Side a is the longest side of the triangle, with a length of 65.0 cm. Determine the measure of $\angle A$ to the nearest tenth of a degree.

Your Turn

Determine the length of side AB in $\triangle ABC$ above, to the nearest tenth of a centimetre.

Answer



$$\angle C = 180° - (\angle A + \angle B)$$
$$\angle C = 180° - (140.6° + 23.0°)$$
$$\angle C = 16.4°$$

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

$$\frac{c}{\sin 16.4°} = \frac{40.0}{\sin 23.0°}$$

$$\frac{c\,\cancel{\sin 23}}{\cancel{\sin 23}} = \frac{40.0 \sin 16.4}{\sin 23}$$

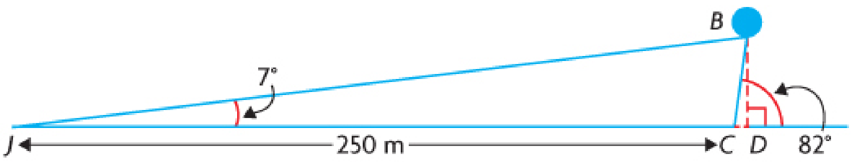
$$c = 28.903\ldots$$

The length of side AB is 28.9 cm, to the nearest tenth of a centimetre.

EXAMPLE 2 Solving a problem using the sine law

Colleen and Juan observed a tethered balloon advertising the opening of a new fitness centre. They were 250 m apart, joined by a line that passed directly below the balloon, and were on the same side of the balloon. Juan observed the balloon at an angle of elevation of 7° while Colleen observed the balloon at an angle of elevation of 82°. Determine the height of the balloon to the nearest metre.

Colleen's Solution



I drew a diagram to represent the situation. The height of the balloon is represented by *BD*. I need to determine the length of *BC* in order to determine the length of *BD*. I can use the sine law in ΔBJC.

$$\angle BCJ = 180° - 82°$$
$$\angle BCJ = 98°$$

I determined the supplement of 82° to determine the measure of a second angle in $\triangle BJC$. This is an obtuse triangle.

$$\angle JBC = 180° - 98° - 7°$$
$$\angle JBC = 75°$$

I determined the measure of the third angle in ΔBJC. This gave me a known side, *JC*, and a known angle opposite this side, $\angle JBC$, in this triangle.

$$\frac{BC}{\sin \angle BJC} = \frac{JC}{\sin \angle JBC}$$

I used the sine law to write an equation that involved *BC* and the known side-angle pair.

$$\frac{BC}{\sin(7°)} = \frac{250}{\sin(75°)}$$
$$BC = \sin(7°)\left(\frac{250}{\sin(75°)}\right)$$
$$BC = 31.542....$$

I substituted the known information into the equation and solved for *BC*.

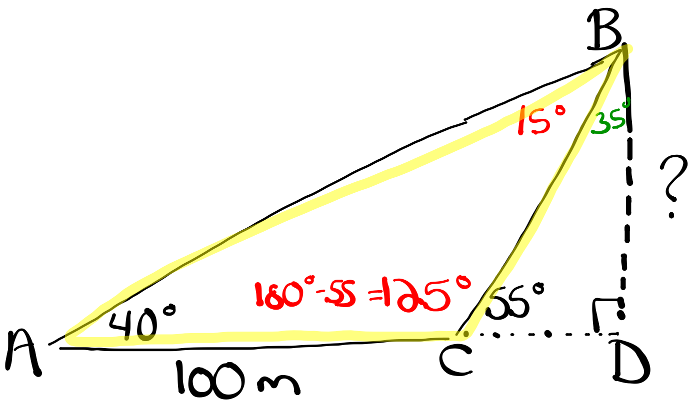
$$\sin(\angle BCD) = \frac{BD}{BC}$$

I wrote an equation that involved *BD*, *BC*, and the known angle in ΔBCD.

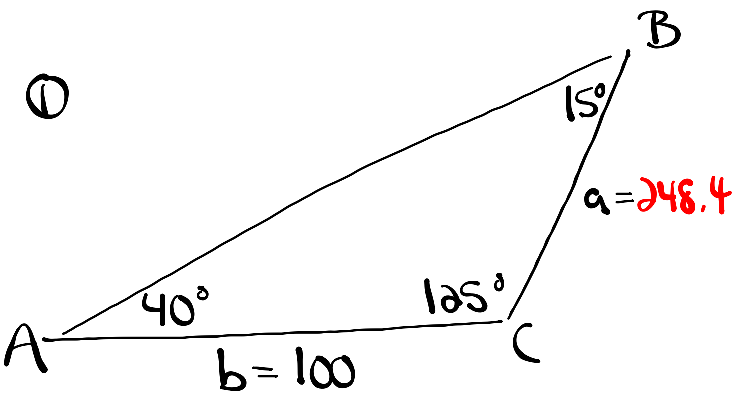
$$\sin(82°) = \frac{BD}{31.542...}$$
$$(31.542...)(\sin(82°)) = BD$$
$$31.235...\text{ m} = BD$$

I substituted the known information into the equation and solved for *BD*.

The advertising balloon is 31 m above the ground.



①



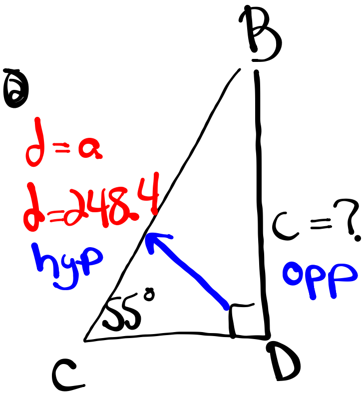
$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

$$\frac{a}{\sin 40^\circ} = \frac{100}{\sin 15^\circ}$$

$$\frac{a\cancel{\sin 15^\circ}}{\cancel{\sin 15^\circ}} = \frac{100\sin 40^\circ}{\sin 15^\circ}$$

$$\boxed{a = 248.4}$$

②



$$\sin\theta = \frac{opp}{hyp}$$

$$\frac{\sin 55^\circ}{1} = \frac{c}{248.4}$$

$$c = 248.4\sin 55^\circ$$

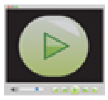
$$\boxed{c = 203.5}$$

EXAMPLE 2 Solving a problem using the sine law

Colleen and Juan observed a tethered balloon advertising the opening of a new fitness centre. They were 250 m apart, joined by a line that passed directly below the balloon, and were on the same side of the balloon. Juan observed the balloon at an angle of elevation of 7° while Colleen observed the balloon at an angle of elevation of 82°. Determine the height of the balloon to the nearest metre.

Your Turn

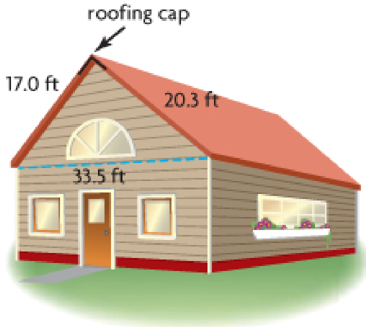
Determine the distance between Juan and the balloon.



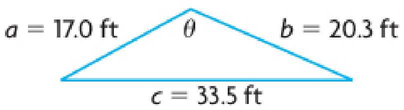
Answer

EXAMPLE 3 Using reasoning and the cosine law to determine the measure of an obtuse angle

The roof of a house consists of two slanted sections, as shown. A roofing cap is being made to fit the crown of the roof, where the two slanted sections meet. Determine the measure of the angle needed for the roofing cap, to the nearest tenth of a degree.



Georgia's Solution: Rearranging the cosine law before substituting



I sketched a triangle to represent the problem situation.

I knew the lengths of all three sides, so I used the cosine law.

Since I wanted to solve for θ, I rearranged the formula to isolate $\cos\theta$.

$$\cos\theta = \frac{a^2 + b^2 - c^2}{2ab}$$

$$\cos\theta = \frac{(17.0)^2 + (20.3)^2 - (33.5)^2}{2(17.0)(20.3)}$$

I substituted the values of a, b, and c into the rearranged formula.

$$\cos\theta = -0.6101\ldots$$

$$\theta = \cos^{-1}(-0.6101\ldots)$$

$$\theta = 127.6039\ldots°$$

The angle for the roofing cap should measure 127.6°.

EXAMPLE 3 Using reasoning and the cosine law to determine the measure of an obtuse angle

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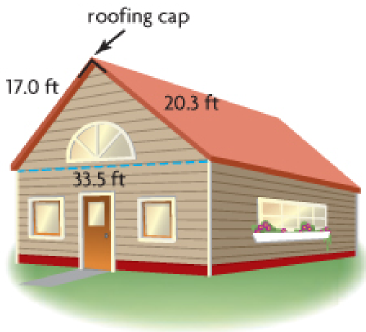
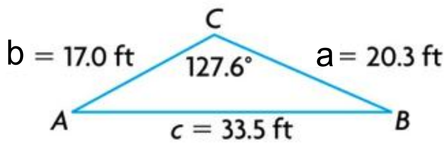


Your Turn

Determine the angle of elevation for each roof section, to the nearest tenth of a degree.

Answer



$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos A = \frac{20.3^2 + 33.5^2 - 17.0^2}{2(20.3)(33.5)}$$

$$\cos A = 0.915...$$

$$\angle A = \cos^{-1}(0.915...)$$

$$\angle A = 23.705...°$$

$$\angle B = 180° - (127.6° + 23.705...°)$$

$$\angle B = 28.694...°$$

$$\angle B = 28.7°$$

The angles of elevation are 23.7° and 28.7°.

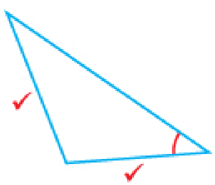
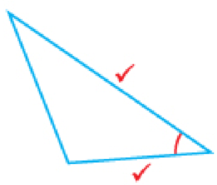
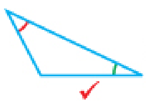
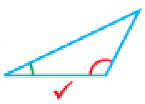
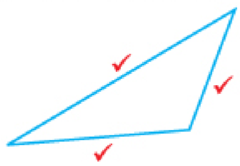
In Summary

Key Idea

- The sine law and cosine law can be used to determine unknown side lengths and angle measures in obtuse triangles.

Need to Know

- The sine law and cosine law are used with obtuse triangles in the same way that they are used with acute triangles.

Use the sine law when you know ...	**Use the cosine law when you know ...**
- the lengths of two sides and the measure of the angle that is opposite a known side 	- the lengths of two sides and the measure of the contained angle 
- the measures of two angles and the length of any side  or 	- the lengths of all three sides 

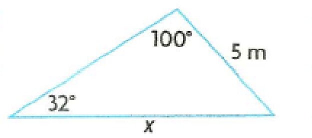
- Be careful when using the sine law to determine the measure of an angle. The inverse sine of a ratio always gives an acute angle, but the supplementary angle has the same ratio. You must decide whether the acute angle, θ, or the obtuse angle, $180° - \theta$, is the correct angle for your triangle.
- Because the cosine ratios for an angle and its supplement are not equal (they are opposites), the measures of the angles determined using the cosine law are always correct.

<u>Assignment:</u> **pgs. 170 - 173**
1, 2, 3, 4, 5, 7, 8, 9, 10, 12

SOLUTIONS ⇒ Ex. 4.2 Proving and Applying the Sine and Cosine Laws for Obtuse Triangles

1. There are errors in each application of the sine law or cosine law.
Identify the errors.

a)

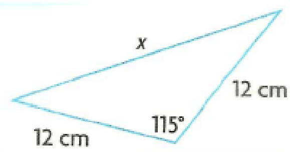


$$\frac{5}{\sin 100°} = \frac{x}{\sin 32°}$$

Correction:

$$\frac{5}{\sin 32°} = \frac{x}{\sin 100°}$$

b)

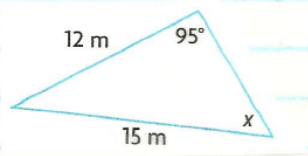


$$\underline{12}^2 = x^2 + 12^2 - 2(12)(\underline{x}) \cos 115°$$

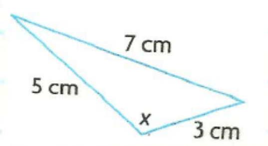
Correction:

$$x^2 = 12^2 + 12^2 - 2(12)(12) \cos 115°$$

2. Which law could be used to determine the unknown angle measure or side length in each triangle? For your answer, choose one of the following: sine law, cosine law, both, neither. Explain your choice.

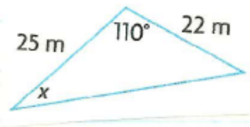
a) 

You would use the sine law since you have 2 "matching pairs."

b) 

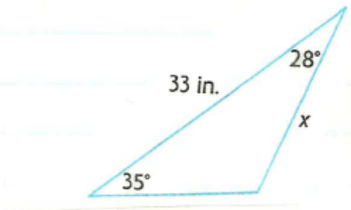
You would use the cosine law since you are given all 3 sides.

c)



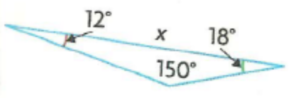
You would use the cosine law since you are given 2 sides and the included angle.

d)



You would use the sine law since you will have 2 "matching pairs" after you find the missing angle.

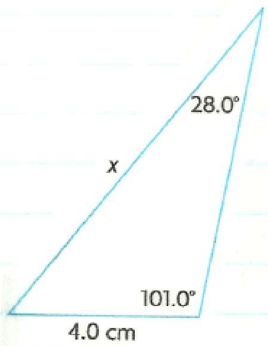
e)



You could not use either the sine law or the cosine law to determine the missing side.

3. Determine the unknown side length in each triangle, to the nearest tenth of a centimeter.

a)

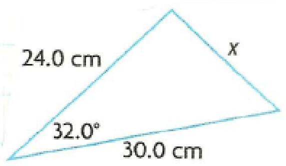


$$\frac{x}{\sin 101°} = \frac{4.0}{\sin 28°}$$

$$\frac{x\cancel{\sin 28°}}{\cancel{\sin 28°}} = \frac{4.0\sin 101°}{\sin 28°}$$

$$x = 8.4 \text{ cm}$$

b)



$$x^2 = (24.0)^2 + (30.0)^2 - 2(24.0)(30.0)\cos 32°$$

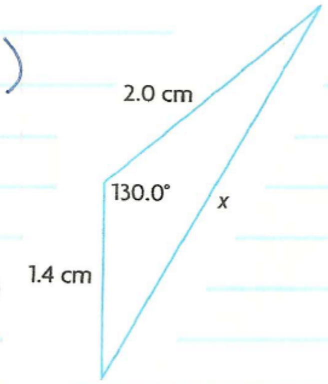
$$x^2 = 576 + 900 - 1440(0.8480)$$

$$x^2 = 1476 - 1221.12$$

$$x^2 = 254.88$$

$$x = 16.0 \text{ cm}$$

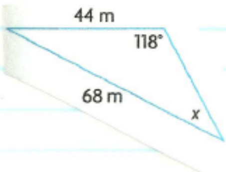
c)



$$x^2 = (2.0)^2 + (1.4)^2 - 2(2.0)(1.4)\cos 130°$$
$$x^2 = 4 + 1.96 - 5.6(-0.6428)$$
$$x^2 = 5.96 + 3.5997$$
$$x^2 = 9.5597$$
$$x = 3.1 \text{ cm}$$

4. Determine the unknown angle measure in each triangle, to the nearest degree.

a)



$$\frac{\sin x}{44} = \frac{\sin 118°}{68}$$

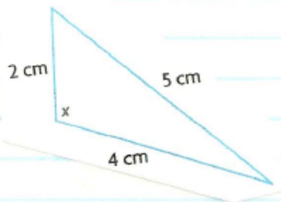
$$\frac{68 \sin x}{68} = \frac{44 \sin 118°}{68}$$

$$\sin x = 0.5713$$

$$x = \sin^{-1}(0.5713)$$

$$x = 35°$$

b)



$$\cos x = \frac{(2)^2 + (4)^2 - (5)^2}{2(2)(4)}$$

$$\cos x = \frac{4 + 16 - 25}{16}$$

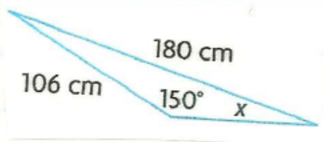
$$\cos x = \frac{-5}{16}$$

$$\cos x = -0.3125$$

$$x = \cos^{-1}(-0.3125)$$

$$x = 108°$$

c)



$$\frac{\sin x}{106} = \frac{\sin 150^\circ}{180}$$

$$\frac{\cancel{180} \sin x}{\cancel{180}} = \frac{106 \sin 150^\circ}{180}$$

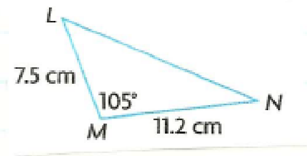
$$\sin x = 0.2944$$

$$x = \sin^{-1}(0.2944)$$

$$x = 17^\circ$$

5. Determine each unknown angle measure to the nearest degree and each unknown side length to the nearest tenth of a centimeter.

a)



$m^2 = (7.5)^2 + (11.2)^2 - 2(7.5)(11.2)\cos 105°$

$m^2 = 56.25 + 125.44 - 168(-0.2588)$

$m^2 = 181.69 + 43.4784$

$m^2 = 225.1684$

$m = 15.0$ cm

$\cos L = \dfrac{(7.5)^2 + (15.0)^2 - (11.2)^2}{2(7.5)(15.0)}$

$\cos L = \dfrac{56.25 + 225 - 125.44}{225}$

$\cos L = \dfrac{155.81}{225}$

$\cos L = 0.6925$

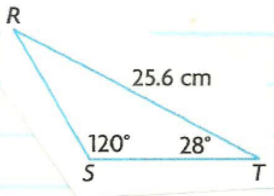
$L = \cos^{-1}(0.6925)$

$L = 46°$

$N = 180° - 105° - 46°$

$N = 29°$

b)



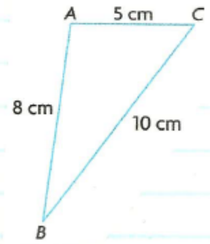
$R = 180° - 120° - 28°$
$R = 32°$

$$\frac{t}{\sin 28°} = \frac{25.6}{\sin 120°} \qquad \frac{r}{\sin 32°} = \frac{25.6}{\sin 120°}$$

$$\frac{t\,\cancel{\sin 120°}}{\cancel{\sin 120°}} = \frac{25.6 \sin 28°}{\sin 120°} \qquad \frac{r\,\cancel{\sin 120°}}{\cancel{\sin 120°}} = \frac{25.6 \sin 32°}{\sin 120°}$$

$t = 13.9\text{ cm}$ $\qquad$ $r = 15.7\text{ cm}$

c)



$$\cos A = \frac{(5)^2 + (8)^2 - (10)^2}{2(5)(8)}$$

$$\cos A = \frac{25 + 64 - 100}{80}$$

$$\cos A = \frac{-11}{80}$$

$$\cos A = -0.1375$$

$$A = \cos^{-1}(-0.1375)$$

$$A = 98°$$

$$\cos B = \frac{(8)^2 + (10)^2 - (5)^2}{2(8)(10)}$$

$$\cos B = \frac{64 + 100 - 25}{160}$$

$$\cos B = \frac{139}{160}$$

$$\cos B = 0.8688$$

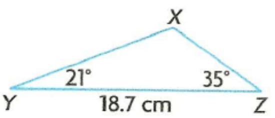
$$B = \cos^{-1}(0.8688)$$

$$B = 30°$$

$$C = 180° - 98° - 30°$$

$$C = 52°$$

d)



$x = 180° - 35° - 21°$

$x = 124°$

$$\frac{y}{\sin 21°} = \frac{18.7}{\sin 124°}$$

$$\frac{y \cancel{\sin 124°}}{\cancel{\sin 124°}} = \frac{18.7 \sin 21°}{\sin 124°}$$

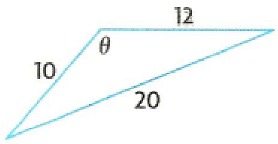
$y = 8.1\text{ cm}$

$$\frac{z}{\sin 35°} = \frac{18.7}{\sin 124°}$$

$$\frac{z \cancel{\sin 124°}}{\cancel{\sin 124°}} = \frac{18.7 \sin 35°}{\sin 124°}$$

$z = 12.9\text{ cm}$

7. Wei-Ting made a mistake when using the cosine law to determine the unknown angle measure below. Identify the cause of the error message on her calculator. Then determine θ to the nearest tenth of a degree.



$20^2 = 10^2 + 12^2 - 2(10)(12)\cos\theta$
$400 = 100 + 144 - 240\cos\theta$
$400 = 244 - 240\cos\theta$
$400 = 4\cos\theta$ ERROR-ORDER OF OPERATIONS!
$100 = \cos\theta$
$\cos^{-1}(100) = \theta$
<error!> $= \theta$

This should be: $400 - 244 = -240\cos\theta$

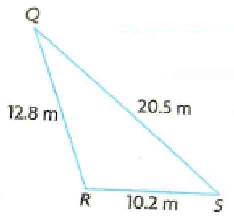
$$\frac{156}{-240} = \frac{\cancel{-240}\cos\theta}{\cancel{-240}}$$

$$-0.6500 = \cos\theta$$

$$\cos^{-1}(-0.6500) = \theta$$

$$131° = \theta$$

8. In $\triangle QRS$, $q = 10.2$ m, $r = 20.5$ m, and $s = 12.8$ m. Solve $\triangle QRS$ by determining the measure of each angle to the nearest tenth of a degree.



$$\cos Q = \frac{(12.8)^2 + (20.5)^2 - (10.2)^2}{2(12.8)(20.5)}$$

$$\cos Q = \frac{163.84 + 420.25 - 104.04}{524.8}$$

$$\cos Q = \frac{480.05}{524.8}$$

$$\cos Q = 0.9147$$

$$Q = \cos^{-1}(0.9147)$$

$$Q = 24°$$

$$\cos S = \frac{(20.5)^2 + (10.2)^2 - (12.8)^2}{2(20.5)(10.2)}$$

$$\cos S = \frac{420.25 + 104.04 - 163.84}{418.2}$$

$$\cos S = \frac{360.45}{418.2}$$

$$\cos S = 0.8619$$

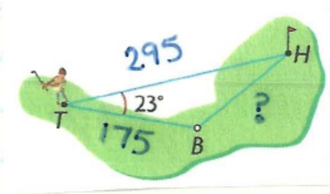
$$S = \cos^{-1}(0.8619)$$

$$S = 30°$$

$$R = 180° - 24° - 30°$$

$$R = 126°$$

9. While golfing, Sahar hits a tee shot from T toward a hole at H. Sahar hits the ball at an angle of 23° to the hole and it lands at B. The scorecard says that H is 295 yd from T. Sahar walks 175 yd to her ball. How far, to the nearest yard, is her ball from the hole?



$t^2 = (295)^2 + (175)^2 - 2(295)(175)\cos 23°$

$t^2 = 87025 + 30625 - 103250(0.9205)$

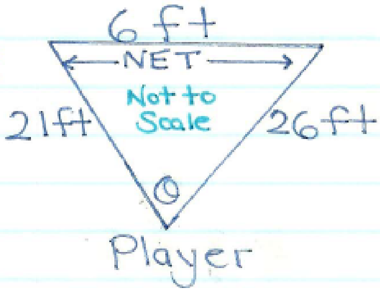
$t^2 = 117650 - 95041.625$

$t^2 = 22608.375$

$t = 150.4$ yd or 150 yd.

10. The posts of a hockey goal are 6 ft apart. A player attempts to score by shooting the puck along the ice from a point that is 21 ft from one post and 26.0 ft from the other post. Within what angle, θ, must the shot be made? Express your answer to the nearest tenth of a degree.

Sketch:



$$\cos\theta = \frac{(21)^2+(26)^2-(6)^2}{2(21)(26)}$$

$$\cos\theta = \frac{441+676-36}{1092}$$

$$\cos\theta = \frac{1081}{1092}$$

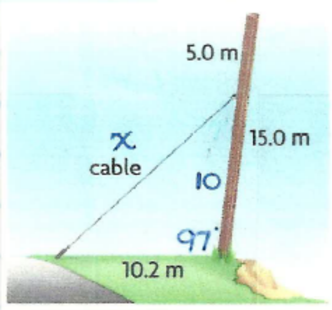
$$\cos\theta = 0.9899$$

$$\theta = \cos^{-1}(0.9899)$$

$$\theta = 8.2^\circ$$

The shot must be made within an 8.2° angle to score from that location.

12. A 15.0 m telephone pole is beginning to lean as the soil erodes. A cable is attached 5.0 m from the top of the pole to prevent the pole from leaning any farther. The cable is secured 10.2 m from the base of the pole. Determine the length of the cable that is needed if the pole is already leaning 7° from the vertical.



$x^2 = (10.2)^2 + (10)^2 - 2(10.2)(10)\cos 97°$

$x^2 = 104.04 + 100 - 204(-0.1219)$

$x^2 = 204.04 + 24.8676$

$x^2 = 228.9076$

$x = 15.1\text{ m}$

FM11-4s2-sine.gsp

FM11-4s2-cosine.gsp

4Ws2e1.mp4

4Ws2e2.mp4

4ws2e3.mp4

4Ws2e4.mp4