1.5 Proofs That Are Not Valid

GOAL

Identify errors in proofs.

EXPLORE...

- Consider the following statement: There are tthree errorss in this sentence. Is the statement valid?

SAMPLE ANSWER

APPLY the Math

EXAMPLE 1 Using reasoning to determine the validity of an argument

①Athletes do not compete in both the Summer and Winter Olympics.
②Hayley Wickenheiser has represented Canada four times at the Winter Olympics.
③Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

Tia read these statements and knew that there was an error. Identify the error in the reasoning.

Tia's Solution

Athletes do not compete in both the Summer and Winter Olympics. — I did some research and found that 18 athletes have competed in both Games. This statement is not valid.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics. — This statement is true. She has played on the national hockey team.

Therefore, Hayley Wickenheiser has not participated in the Summer Olympics. — The conclusion was false because the first statement was false. Hayley played for Canada in the softball competition in the 2000 Summer Olympics.

EXAMPLE 1 Using reasoning to determine the validity of an argument

Athletes do not compete in both the Summer and Winter Olympics.
Hayley Wickenheiser has represented Canada four times at the Winter Olympics.
Therefore, Hayley Wickenheiser has not participated in the Summer Olympics.

Tia read these statements and knew that there was an error. Identify the error in the reasoning.

Hayley Wickenheiser has represented Canada four times at the Winter Olympics.

Your Turn

Zack is a high school student. All high school students dislike cooking. Therefore, Zack dislikes cooking. Where is the error in the reasoning?

Communication ***Tip***

Stereotypes are generalizations based on culture, gender, religion, or race. There are always counterexamples to stereotypes, so conclusions based on stereotypes are not valid.

Answer

EXAMPLE 2 Using reasoning to determine the validity of a proof

Bev claims he can prove that $3 = 4$.

Bev's Proof

Suppose that: $a + b = c$

This statement can be written as: $4a - 3a + 4b - 3b = 4c - 3c$

After reorganizing, it becomes: $4a + 4b - 4c = 3a + 3b - 3c$

Using the distributive property, $4(a + b - c) = 3(a + b - c)$ ← factor

Dividing both sides by $(a + b - c)$, $4 = 3$

Show that Bev has written an **invalid proof**.

invalid proof
A proof that contains an error in reasoning or that contains invalid assumptions.

Pru's Solution

Suppose that:

$$a + b = c \checkmark$$

Bev's **premise** was made at the beginning of the proof. Since variables can be used to represent any numbers, this part of the proof is valid.

premise
A statement assumed to be true.

$$4a - 3a + 4b - 3b = 4c - 3c \checkmark$$

Bev substituted $4a - 3a$ for a since $4a - 3a = a$.
Bev substituted $4b - 3b$ for b since $4b - 3b = b$.
Bev substituted $4c - 3c$ for c since $4c - 3c = c$.

$$4a + 4b - 4c = 3a + 3b - 3c \checkmark$$

I reorganized the equation and I came up with the same result that Bev did when he reorganized. Simplifying would take me back to the premise. This part of the proof is valid.

$$4(a + b - c) = 3(a + b - c) \checkmark$$

Since each side of the equation has the same coefficient for all the terms, factoring both sides is a valid step.

$$\frac{4(a + b - c)}{(a + b - c)} = \frac{3(a + b - c)}{(a + b - c)}$$

This step appears to be valid, but when I looked at the divisor, I identified the flaw.

$$a + b = c$$
$$a + b - c = c - c$$
$$a + b - c = 0$$

When I rearranged the premise, I determined that the divisor equalled zero.

Dividing both sides of the equation by $a + b - c$ is not valid. Division by zero is undefined.

EXAMPLE 2 Using reasoning to determine the validity of a proof

Bev claims he can prove that $3 = 4$.

Bev's Proof

Suppose that: $a + b = c$

This statement can be written as: $4a - 3a + 4b - 3b = 4c - 3c$

After reorganizing, it becomes: $4a + 4b - 4c = 3a + 3b - 3c$

Using the distributive property, $4(a + b - c) = 3(a + b - c)$

Dividing both sides by $(a + b - c)$, $4 = 3$

Show that Bev has written an **invalid proof**.

Your Turn

How could this type of false proof be used to suggest that $65 = 64$?

Answer

EXAMPLE 3 Using reasoning to determine the validity of a proof

Liz claims she has proved that $-5 = 5$.

Liz's Proof

I assumed that $-5 = 5$.
Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$
This means that my assumption, $-5 = 5$, must be correct.

Where is the error in Liz's proof?

Simon's Solution

I assumed that $-5 = 5$.

Liz started off with the false assumption that the two numbers were equal.

Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$

Everything that comes after the false assumption doesn't matter because the reasoning is built on the false assumption.

Even though $25 = 25$, the underlying premise is not true.

$-5 \neq 5$

Liz's conclusion is built on a false assumption, and the conclusion she reaches is the same as her assumption.

If an assumption is not true, then any argument that was built on the assumption is not valid.

Circular reasoning has resulted from these steps. Starting with an error and then ending by saying that the error has been proved is arguing in a circle.

circular reasoning
An argument that is incorrect because it makes use of the conclusion to be proved.

EXAMPLE 3 Using reasoning to determine the validity of a proof

Liz claims she has proved that $-5 = 5$.

Liz's Proof
I assumed that $-5 = 5$.
Then I squared both sides: $(-5)^2 = 5^2$
I got a true statement: $25 = 25$
This means that my assumption, $-5 = 5$, must be correct.

Where is the error in Liz's proof?

Your Turn

How is an error in a premise like a counterexample?

Answer

EXAMPLE 4 Using reasoning to determine the validity of a proof

Hossai is trying to prove the following number trick:
Choose any number. Add 3. Double it. Add 4. Divide by 2. Take away the number you started with.

Each time Hossai tries the trick, she ends up with 5. Her proof, however, does not give the same result.

Hossai's Proof

n	Choose any number.
$n + 3$	Add 3.
$2n + 6$	Double it.
$2n + 10$	Add 4.
$2n + 5$	Divide by 2.
$n + 5$	Take away the number you started with.

Where is the error in Hossai's proof?

Sheri's Solution

$1 \longrightarrow 5$
$10 \longrightarrow 5$

I tried the number trick twice, for the number 1 and the number 10. Both times, I ended up with 5. The math trick worked for Hossai and for me, so the error must be in Hossai's proof.

n ✔ — The variable n can represent any number. This step is valid.

$n + 3$ ✔ — Adding 3 to n is correctly represented.

$2n + 6$ ✔ — Doubling a quantity is multiplying by 2. This step is valid. Its simplification is correct as well.

$2n + 10$ ✔ — Adding 4 to the expression is correctly represented, and the simplification is correct.

$2n + 5$ ✘ — The entire expression should be divided by 2, not just the constant. This step is where the mistake occurred.

I corrected the mistake:

$$\frac{2n + 10}{2} = n + 5$$

$$n + 5 - n = 5$$

I completed Hossai's proof by subtracting n. I showed that the answer will be 5 for any number.

EXAMPLE 4 Using reasoning to determine the validity of a proof

Hossai is trying to prove the following number trick:
Choose any number. Add 3. Double it. Add 4. Divide by 2. Take away the number you started with.

Each time Hossai tries the trick, she ends up with 5. Her proof, however, does not give the same result.

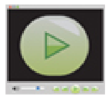
Hossai's Proof

n	Choose any number.
$n + 3$	Add 3.
$2n + 6$	Double it.
$2n + 10$	Add 4.
$2n + 5$	Divide by 2.
$n + 5$	Take away the number you started with.

Where is the error in Hossai's proof?

Your Turn

Is there a number that will not work in Hossai's number trick? Explain.



Answer

In Summary

Key Idea

- A single error in reasoning will break down the logical argument of a deductive proof. This will result in an invalid conclusion, or a conclusion that is not supported by the proof.

Need to Know

- Division by zero always creates an error in a proof, leading to an invalid conclusion.
- Circular reasoning must be avoided. Be careful not to assume a result that follows from what you are trying to prove.
- The reason you are writing a proof is so that others can read and understand it. After you write a proof, have someone else who has not seen your proof read it. If this person gets confused, your proof may need to be clarified.

Assignment: pages 42-44

Questions: 1, 2, 3, 5, 6ab, 7

SOLUTIONS ⇒ 1.5 Proofs That Are Not Valid

1. Determine the error in each example of deductive reasoning.

a) All runners train on a daily basis. Gabriel is a runner. Therefore, Gabriel trains daily.

⇒ The statement "all runners train on a daily basis" is invalid.

b) All squares have four right angles. Quadrilateral PQRS has four right angles. Therefore, PQRS is a square.

⇒ The reasoning leading to the conclusion is invalid. Rectangles also have four right angles.

2. According to this proof, 5=7. Identify the error.

Proof

$1 = 1 + 1$

$2(1) = 2(1+1)$

$2(1) + 3 = 2(1+1) + 3$

$2 + 3 = 4 + 3$

$5 = 7$

Error ⇒ The first line of the proof is invalid.

3. Mickey says he can prove that 2=0. Here is his proof.

Let both a and b be equal to 1.

$a = b$ ✓	Transitive property
$a^2 = b^2$ ✓	Squaring both sides
$a^2 - b^2 = 0$ ✓	Subtracting b^2 from both sides
$(a - b)(a + b) = 0$ ✓	Factoring a difference of squares
$\frac{(a - b)(a + b)}{(a - b)} = \frac{0}{(a - b)}$ ✗	Dividing both sides by $a - b$
$1(a + b) = 0$	Simplifying
$a + b = 0$	
$1 + 1 = 0$	Substitution
$2 = 0$	

Explain whether each statement in Mickey's proof is valid.

Line 5 is not valid since Mickey is dividing by $(a-b)$ and $a-b=0$.

5. Ali created a math trick in which she always ended with 4. When Ali tried to prove her trick, however, it did not work.

Ali's Proof

n ✓	I used n to represent any number.
$2n$ ✓	Multiply by 2.
$2n + 8$ ✓	Add 8.
$2n + 4$ ✗	Divide by 2.
$n + 4$	Subtract your starting number.

Identify the error in Ali's proof, and explain why her reasoning is incorrect.

Error ⇒ Ali did not correctly divide by 2 in line 4.

$$\left\{ \frac{2n+8}{2} = n+4 \right\}$$

6. Connie tried this number trick:

- Write down the number of your street address.
- Multiply by 2.
- Add the number of days in a week.
- Multiply by 50.
- Add your age.
- Subtract the number of days in a year.
- Add 15.

Connie's result was a number in which the tens and ones digits were her age and the rest of the digits were the number from her street address. She tried to prove why this works, but her final expression did not make sense.

a) Try this number trick to see if you get the same result as Connie.

↳ Street Address: 14 Age: 16

n	14
x2	28
+7	35
x50	1750
+16	1766
-365	1401
+15	1416

Let n represent any house number.

$2n$	Multiply by 2.
$2n + 7$	Add the number of days in a week.
$100n + 350$	Multiply by 50.
Let a represent any age.	
$100n + 350 + a$	Add your age.
$100n + 350 + a - 360$	Subtract the number of days in a year.
$100n + a + 5$	Add 15.

b) Determine the errors in her proof, and then correct them.

⇒ There is an error in line 6 of Connie's proof. She subtracted 360 instead of 365 for the number of days in the year.

Correction.

Line 6: $100n + 350 + a - 365$

Line 7: $100n + a - 15 + 15$

↳ $100n + a$

7. According to this proof, 2=1. Determine the error in reasoning.
Let $a = b$.

$a^2 = ab$ ✓	Multiply by a.
$a^2 + a^2 = a^2 + ab$ ✓	Add a^2.
$2a^2 = a^2 + ab$ ✓	Simplify.
$2a^2 - 2ab = a^2 + ab - 2ab$ ✓	Subtract $2ab$.
$2a^2 - 2ab = a^2 - ab$ ✓	Simplify.
$2(a^2 - ab) = 1(a^2 - ab)$ ✓	Factor.
$2 = 1$ ✗	Divide by $(a^2 - ab)$.

Error ⇒ In the final line, there is a division by 0.

↳ $a^2 - ab$
$= a(a-b)$
$= a(0)$ (since $a=b$)
$= 0$

PM11-1s5.gsp

1s5e1 finalt.mp4

1s5e2 finalt.mp4

1s5e3 finalt.mp4

1s5e4 finalt.mp4

1s5e5 finalt.mp4