

## Warm Up

2. Factor each of the following:

$$x^{27} - 1 \qquad (x^2 + 1)^{\frac{1}{2}} + 3(x^2 + 1)^{-\frac{1}{2}}$$

$$\underline{(x^9 - 1)}(x^{18} + x^9 + 1)$$

$$\underline{(x^3 - 1)}(x^6 + x^3 + 1)(x^{18} + x^9 + 1)$$

$$(x - 1)(x^2 + x + 1)(x^6 + x^3 + 1)(x^{18} + x^9 + 1)$$

$$\underline{(x^2 + 1)^{\frac{1}{2}}} + 3\underline{(x^2 + 1)^{-\frac{1}{2}}}$$

$$(x^2 + 1)^{-\frac{1}{2}}(x^2 + 1 + 3)$$

$$(x^2 + 1)^{-\frac{1}{2}}(x^2 + 4)$$

$$\frac{(x^2 + 1)^{\frac{1}{2}}}{(x^2 + 1)^{-\frac{1}{2}}} = (x^2 + 1)^1$$

$$\frac{3(x^2 + 1)^{-\frac{1}{2}}}{(x^2 + 1)^{-\frac{1}{2}}} = 3(x^2 + 1)^0 = 3$$

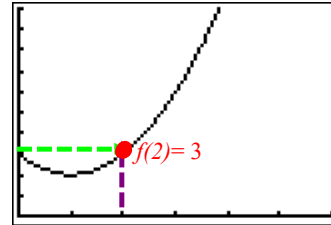
# Limit of a Function

Let's examine the function  $f(x) = x^2 - 2x + 3$  (parabola)

	Plot2	Plot3
Y1	$x^2 - 2x + 3$	
Y2	=	
Y3	=	
Y4	=	
Y5	=	
Y6	=	
Y7	=	

X	Y1
0	3
1	2
2	3
3	6
4	11
5	18
6	27

X=0



We can see that  $f(2) = 3$ ...let's check the behaviour of  $f$  as we get closer and closer to  $x = 2$ .

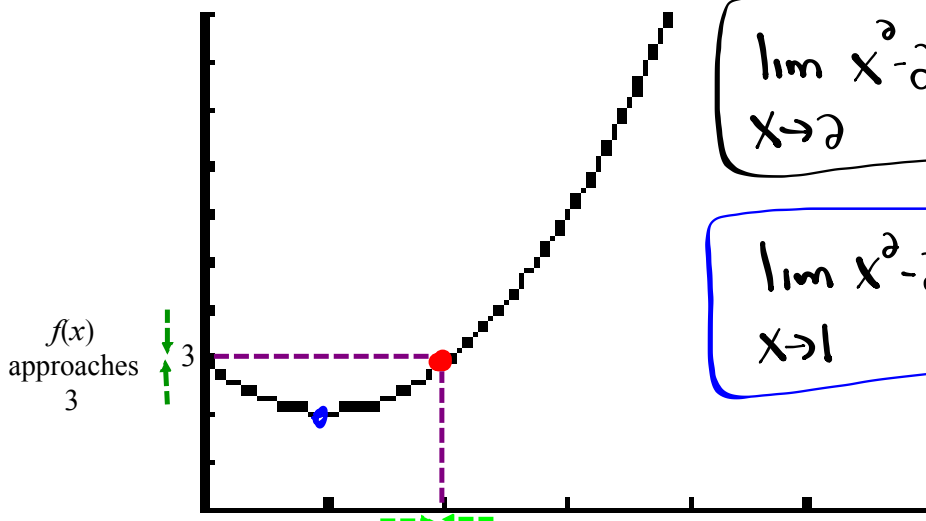
X	Y1
1.9	2.7225
1.95	2.8025
2.05	2.1025
2.1	2.21
2.15	2.3225

X=1.85

← As  $x$  gets closer to 2 from the left  $y$  is getting closer to 3.

← As  $x$  gets closer to 2 from the right  $y$  is getting closer to 3.

From the above, the notion of the limit of a function arises...



$$\lim_{x \rightarrow 2} x^2 - 2x + 3 = \underline{\underline{3}}$$

height

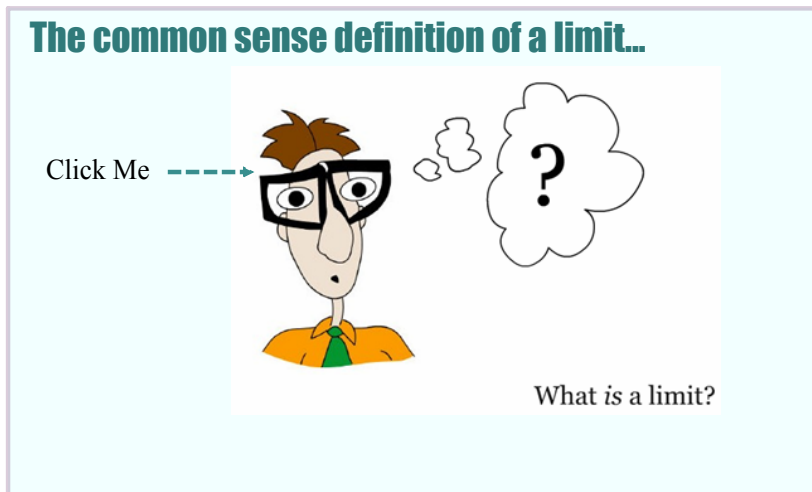
$$\lim_{x \rightarrow 1} x^2 - 2x + 3 = \underline{\underline{2}}$$

height

As  $x$  approaches 2

**Notation:**  $\lim_{x \rightarrow 2} f(x) = 3$

"The limit of the function  $f(x)$  as  $x$  approaches 2 is equal to 3."



## A formal definition of a limit...

We write  $\lim_{x \rightarrow a} f(x) = L$  if we can make the

values of  $f(x)$  arbitrarily close to  $L$

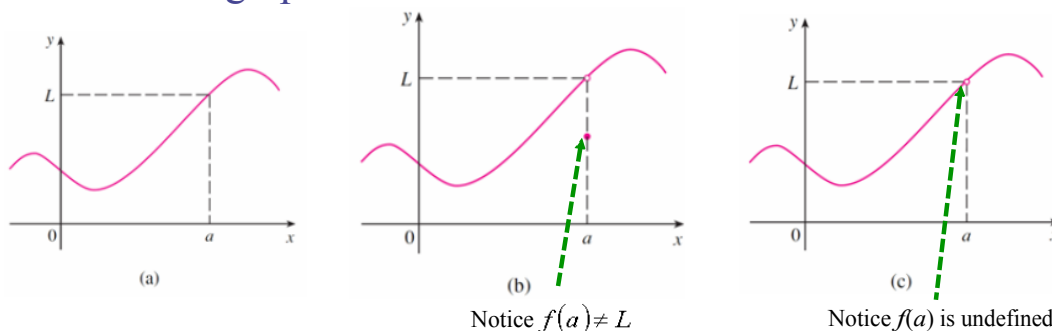
- (as close to  $L$  as we like)

by taking  $x$  to be sufficiently close to  $a$

- (on either side of  $a$ )

but not equal to  $a$ .

Look at the graphs of these three functions...



But in each case, regardless of what happens at  $a$ , it is true that

$$\lim_{x \rightarrow a} f(x) = L$$

# Evaluating Limits

## I. Using a Graph:

- We looked at this in the previous two examples

## II. Algebraically:

- Direct Substitution...

Examples:

$$\lim_{x \rightarrow -2} \frac{x^2 - 2x + 1}{x + 3}$$

$$\lim_{x \rightarrow -2} \frac{(-2)^2 - 2(-2) + 1}{(-2) + 3} = \frac{9}{1}$$

(y-value)  
height

$$\lim_{x \rightarrow 3} (16 - x^2)$$

$$\lim_{x \rightarrow 3} (16 - (3)^2) = 7$$

(y-value)  
height

- Indeterminate limits...  $\Rightarrow$  Direct substitution leads to  $\frac{0}{0}$

- $\Rightarrow$  Factor
- $\Rightarrow$  Rationalize
- $\Rightarrow$  Expand
- $\Rightarrow$  Find Common Denominators

Examples:

$$\lim_{x \rightarrow 4} \frac{x^2 - 16}{x - 4}$$

$$\lim_{x \rightarrow 4} \frac{(x+4)\cancel{(x-4)}}{\cancel{x-4}}$$

$$\lim_{x \rightarrow 4} (4 + 4) = \underline{8}$$

(y-value)  
height

$$\lim_{h \rightarrow 0} \frac{(\sqrt{4+h}-2)(\sqrt{4+h}+2)}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \rightarrow 0} \frac{4+h-4}{h(\sqrt{4+h}+2)}$$

$$\lim_{h \rightarrow 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{4+h}+2)}$$

$$\lim_{h \rightarrow 0} \frac{1}{(\sqrt{4+0}+2)} = \left(\frac{1}{4}\right)$$

Try these...remember to use your algebra skills to try and eliminate the indeterminate form.

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\underline{(x+2)^2} - \underline{(x-2)^2}}$$

← common factor
← diff squares

$$\lim_{x \rightarrow -2} \frac{\underline{x^4} - \underline{16}}{x^3 + 8}$$

← diff squares
← sum cubes

$$\lim_{x \rightarrow 0} \frac{x(x+3)}{(\underline{x+2})(\underline{x-2})(\underline{x+2})(\underline{x-2})}$$

$$\lim_{x \rightarrow 2} \frac{(\underline{x^2+4})(\underline{x^2-4})}{(x+2)(x^2-2x+4)}$$

← diff squares

$$\lim_{x \rightarrow 0} \frac{x(x+3)}{(x+2)(x-2)(x+2)(x-2)}$$

$$\lim_{x \rightarrow 2} \frac{(\underline{x^2+4})(\cancel{x+2})(\cancel{x-2})}{(x+2)(\underline{x^2-2x+4})}$$

$$\lim_{x \rightarrow 0} \frac{x(x+3)}{(2x)(4)}$$

$$= \frac{(8)(-4)}{12} = \frac{-32}{12} = \left(\frac{-8}{3}\right)$$

$$\lim_{x \rightarrow 0} \frac{\cancel{x}(\cancel{x+3})}{8\cancel{x}} = \left(\frac{3}{8}\right)$$

$$\lim_{x \rightarrow 2} \frac{\underline{(x+2)}^2 - 16}{\underline{x^2} - 4}$$

$$\lim_{x \rightarrow 2} \frac{\underline{(x+2+4)}(\underline{x+2-4})}{(\underline{x+2})(\underline{x-2})}$$

$$\lim_{x \rightarrow 2} \frac{\underline{(x+6)}(\underline{x-2})}{(\underline{x+2})(\underline{x-2})} = \frac{8}{4} = 2$$

$$\lim_{x \rightarrow 2} \frac{\overset{2x}{x} - \frac{1}{\underset{2x}{2}}}{(x-2)2x}$$

$$\lim_{x \rightarrow 2} \frac{\overset{-1}{\cancel{2-x}}}{\underline{2x}(\underline{x-2})} = \frac{-1}{4}$$

# Homework

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{(x+2)^2 - (x-2)^2} \quad \leftarrow \text{expanding}$$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{(x+2)(x+2) - (x-2)(x-2)}$$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{x^2 + 4x + 4 - (x^2 - 4x + 4)}$$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{\cancel{x^2} + 4x + 4 - \cancel{x^2} + 4x - 4}$$

$$\lim_{x \rightarrow 0} \frac{x^2 + 3x}{8x} \quad \leftarrow \text{common factor}$$

$$\lim_{x \rightarrow 0} \frac{\cancel{x}(x+3)}{8\cancel{x}} = \left(\frac{3}{8}\right)$$