

Inverse of a Relation

An inverse function is a second function which undoes the work of the first one.

1. Introduction

Suppose we have a function f that takes x to y , so that

$$f(x) = y.$$

An inverse function, which we call f^{-1} , is another function that takes y back to x . So

$$f^{-1}(y) = x.$$

For f^{-1} to be an inverse of f , this needs to work for every x that f acts upon.

Inverse of a Relation

The inverse of a relation is found by interchanging the x -coordinates and y -coordinates of the ordered pairs of the relation. In other words, for every ordered pair (x, y) of a relation, there is an ordered pair (y, x) on the inverse of the relation. This means that the graphs of a relation and its inverse are reflections of each other in the line $y = x$.

$(x, y) \rightarrow (y, x)$ In plain English....the x and y coordinates will just switch places

The inverse of a function $y = f(x)$ may be written in the form $x = f(y)$. The inverse of a function is not necessarily a function. When the inverse of f is itself a function, it is denoted as f^{-1} and read as "f inverse." When the inverse of a function is not a function, it may be possible to restrict the domain to obtain an inverse function for a portion of the original function.

The inverse of a function reverses the processes represented by that function. Functions $f(x)$ and $g(x)$ are inverses of each other if the operations of $f(x)$ reverse all the operations of $g(x)$ in the opposite order and the operations of $g(x)$ reverse all the operations of $f(x)$ in the opposite order.

For example, $f(x) = 2x + 1$ multiplies the input value by 2 and then adds 1. The inverse function subtracts 1 from the input value and then divides by 2. The inverse function is $f^{-1}(x) = \frac{x-1}{2}$.

$$f(x) = 2x + 1$$

| | |
|------|------|
| x | y |
| -3 | -5 |

(A red arrow points from the x in the table to the x in the equation above.)

$$f^{-1}(x) = \frac{x-1}{2}$$

| | |
|------|------|
| x | y |
| -5 | -3 |

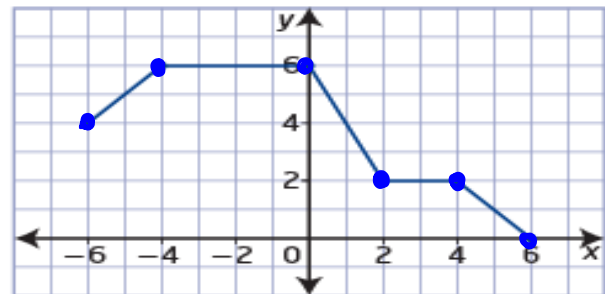
(A blue arrow points from the x in the table to the x in the equation above.)

Example 1

Graph an Inverse

Consider the graph of the relation shown.

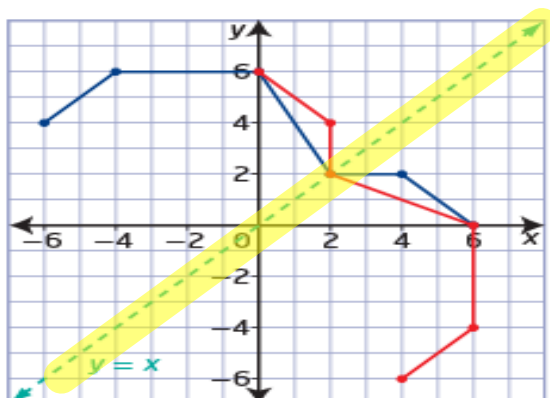
- Sketch the graph of the inverse relation.
- State the domain and range of the relation and its inverse.
- Determine whether the relation and its inverse are functions.



Solution

- To graph the inverse relation, interchange the x -coordinates and y -coordinates of key points on the graph of the relation.

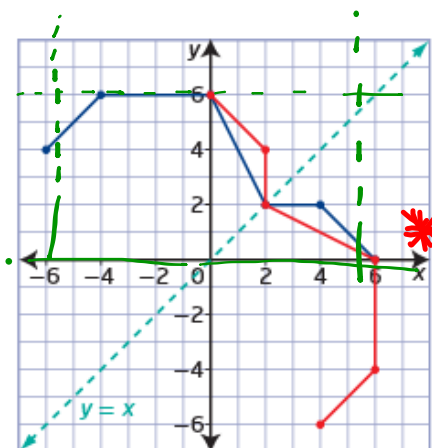
| Points on the Relation | Points on the Inverse Relation |
|------------------------|--------------------------------|
| $(-6, 4)$ | $(4, -6)$ |
| $(-4, 6)$ | $(6, -4)$ |
| $(0, 6)$ | $(6, 0)$ |
| $(2, 2)$ | $(2, 2)$ |
| $(4, 2)$ | $(2, 4)$ |
| $(6, 0)$ | $(0, 6)$ |



The graphs are reflections of each other in the line $y = x$. The points on the graph of the relation are related to the points on the graph of the inverse relation by the mapping $(x, y) \rightarrow (y, x)$.

What points are invariant after a reflection in the line $y = x$?

b) State the domain and range of the relation and its inverse.



| | Domain | Range |
|------------------|---|---|
| Relation | $\{x \mid -6 \leq x \leq 6, x \in \mathbb{R}\}$ | $\{y \mid 0 \leq y \leq 6, y \in \mathbb{R}\}$ |
| Inverse Relation | $\{x \mid 0 \leq x \leq 6, x \in \mathbb{R}\}$ | $\{y \mid -6 \leq y \leq 6, y \in \mathbb{R}\}$ |

$[-6, 6]$

$[0, 6]$

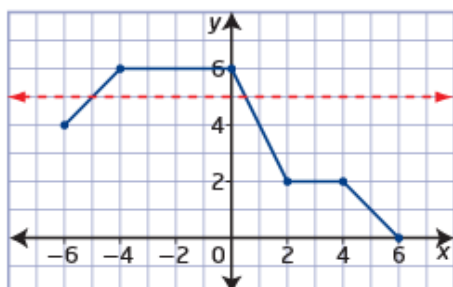
$[0, 6]$

$[-6, 6]$

The domain of the relation becomes the range of the inverse relation and the range of the relation becomes the domain of the inverse relation.

In plain English....the x and y coordinates will just switch places

c) Determine whether the relation and its inverse are functions.



horizontal line test

- a test used to determine if the graph of an inverse relation will be a function
- if it is possible for a horizontal line to intersect the graph of a relation more than once, then the inverse of the relation is not a function

The inverse relation is not a function of x because it fails the vertical line test. There is more than one value of y in the range for at least one value of x in the domain. You can confirm this by using the **horizontal line test** on the graph of the original relation.

Example 2

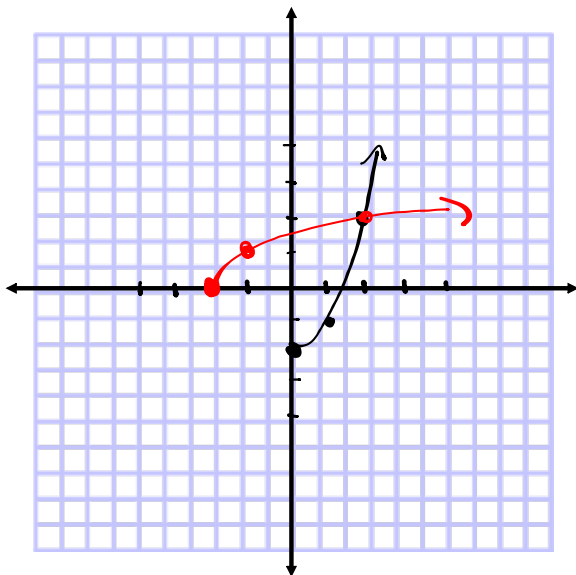
Restrict the Domain

Consider the function $f(x) = x^2 - 2$.

(Parabola)

- Graph the function $f(x)$. Is the inverse of $f(x)$ a function?
- Graph the inverse of $f(x)$ on the same set of coordinate axes.
- Describe how the domain of $f(x)$ could be restricted so that the inverse of $f(x)$ is a function.

No (f(x) = x^2 - 2 would fail the HLT)



$(x, y) \rightarrow (y, x)$

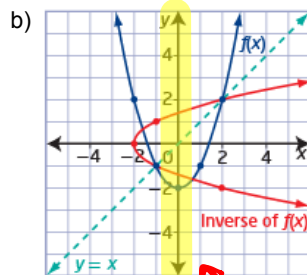
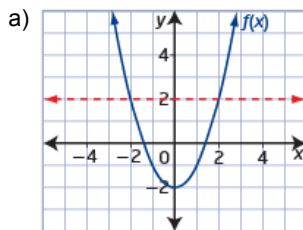
$f(x) = x^2 - 2$

| x | y |
|----|----|
| 0 | -2 |
| -1 | -1 |
| 2 | 2 |

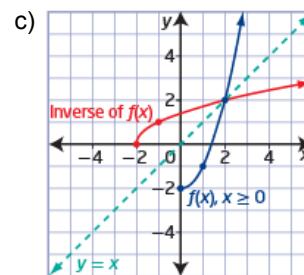
Inverse

| x | y |
|----|---|
| -2 | 0 |
| -1 | 1 |
| 2 | 2 |

Solutions



x=0



- c) The inverse of $f(x)$ is a function if the graph of $f(x)$ passes the horizontal line test.

One possibility is to restrict the domain of $f(x)$ so that the resulting graph is only one half of the parabola.

Since the equation of the axis of symmetry is $x = 0$, restrict the domain to $\{x \mid x \geq 0, x \in \mathbb{R}\}$.

Ex. if axis of sym. is $x = -4$ restrict domain to $\{x \mid x \geq -4, x \in \mathbb{R}\}$

Example 3

Determine the Equation of the Inverse

Algebraically determine the equation of the inverse of each function.

Verify graphically that the relations are inverses of each other.

a) $f(x) = 3x + 6$ (Linear) \rightarrow would pass HLT

b) $f(x) = x^2 - 4$

a) $f(x) = 3x + 6$

① $y = 3x + 6$

② $x = 3y + 6$

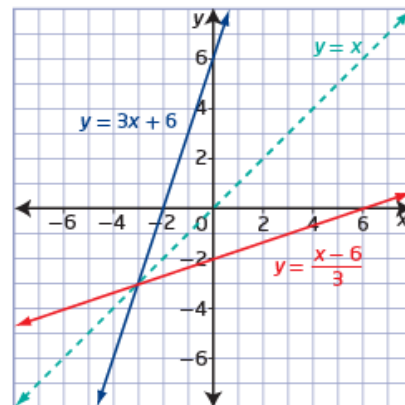
③ $\frac{x-6}{3} = \frac{3y}{3}$

$\frac{x-6}{3} = y$

④ $f^{-1}(x) = \frac{x-6}{3} = \frac{x}{3} - 2$

- 1) Replace $f(x)$ with y .
- 2) Switch x 's and y 's.
- 3) Solve for y .
- 4) Replace y with $f^{-1}(x)$.
(if the inverse is a function!)

Graph $y = 3x + 6$ and $y = \frac{x-6}{3}$ on the same set of coordinate axes.



Determine the Equation of the Inverse

b) $f(x) = x^2 - 4$ (Parabola) \rightarrow would fail the HLT

- 1) Replace $f(x)$ with y .
- 2) Switch x 's and y 's.
- 3) Solve for y .
- 4) Replace y with $f^{-1}(x)$.
(if the inverse is a function!)

① $y = x^2 - 4$

② $x = y^2 - 4$

③ $x + 4 = y^2$

$\pm \sqrt{x+4} = y$

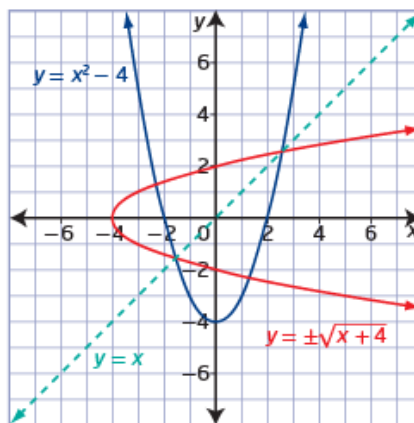
$y = \pm \sqrt{x+4}$

Why is this y not replaced with $f^{-1}(x)$? What could be done so that $f^{-1}(x)$ could be used?

restrict domain of $f(x) \rightarrow \{x \mid x \geq 0, x \in \mathbb{R}\}$

Graph $y = x^2 - 4$ and $y = \pm\sqrt{x+4}$ on the same set of coordinate axes.

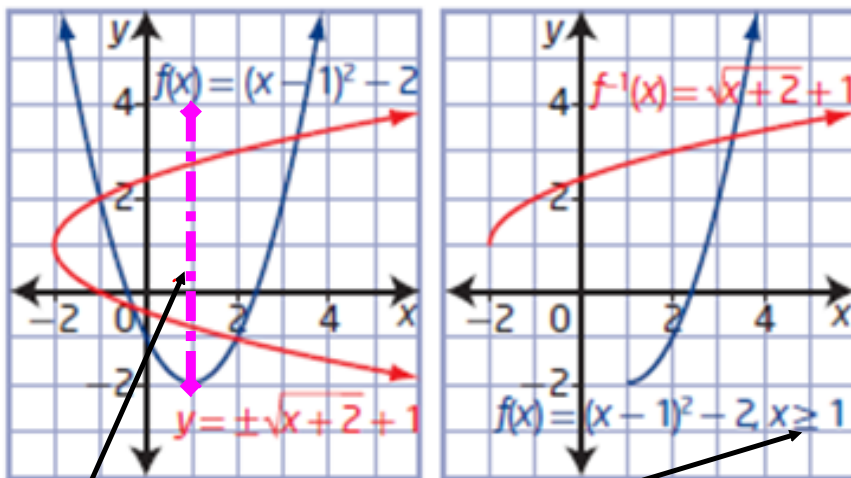
$f^{-1}(x) = \sqrt{x+4}$



Another example of how to restrict the domain

f) $y = \pm\sqrt{x+2} + 1$

restricted domain
 $\{x \mid x \geq 1, x \in \mathbb{R}\}$



axis of symmetry:
 $x=1$

restrict domain to
 $\{x \mid x \geq 1, x \in \mathbb{R}\}$

Inverse of a Relation

Key Ideas

- You can find the inverse of a relation by interchanging the x -coordinates and y -coordinates of the graph.
- The graph of the inverse of a relation is the graph of the relation reflected in the line $y = x$.
- The domain and range of a relation become the range and domain, respectively, of the inverse of the relation.
- Use the horizontal line test to determine if an inverse will be a function.
- You can create an inverse that is a function over a specified interval by restricting the domain of a function.
- When the inverse of a function $f(x)$ is itself a function, it is denoted by $f^{-1}(x)$.
- You can verify graphically whether two functions are inverses of each other.

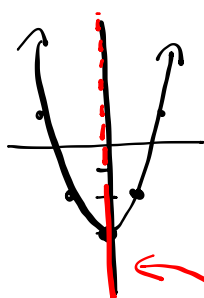
Homework

Practice Problems...

Pages 51 - 55

#2, 3, 5, 6, 8, 9, 11, 15, 18, 20, 21

$$f(x) = x^2 - 3$$



axis of symmetry $x=0$
restrict domain of $f(x)$
to $\{x|x \geq 0, x \in \mathbb{R}\}$

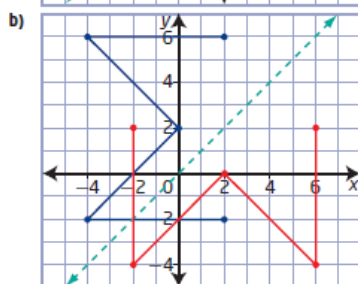
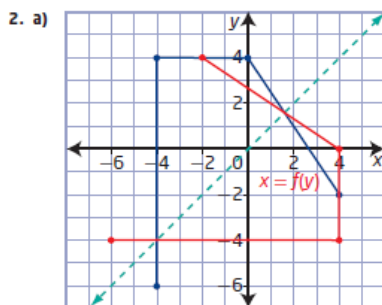
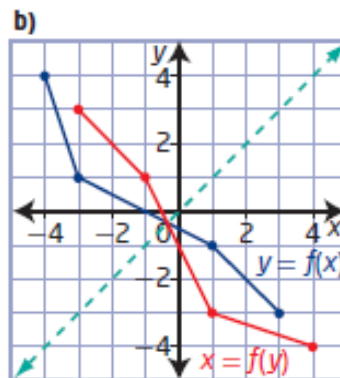
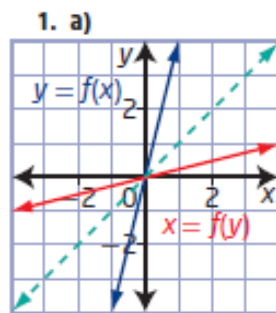
What if given the function algebraically?

Determine algebraically the equation of the inverse of each function.

a) $f(x) = 3x - 6$ **b)** $f(x) = \frac{1}{2}x + 5$

c) $f(x) = \frac{1}{3}(x + 12)$ **d)** $f(x) = \frac{8x + 12}{4}$

1.4 Inverse of a Relation, pages 51 to 55

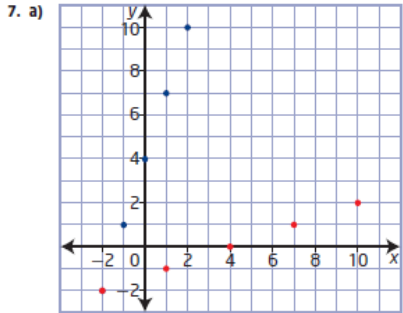


3. a) The graph is a function but the inverse will be a relation.
 b) The graph and its inverse are functions.
 c) The graph and its inverse are relations.

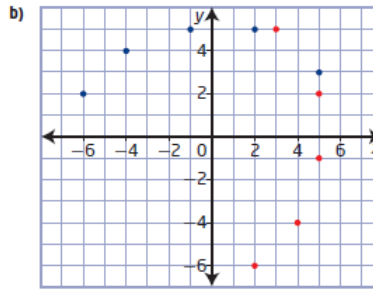
4. Examples:

- a) $\{x \mid x \geq 0, x \in \mathbb{R}\}$ or $\{x \mid x \leq 0, x \in \mathbb{R}\}$
 b) $\{x \mid x \geq -2, x \in \mathbb{R}\}$ or $\{x \mid x \leq -2, x \in \mathbb{R}\}$
 c) $\{x \mid x \geq 4, x \in \mathbb{R}\}$ or $\{x \mid x \leq 4, x \in \mathbb{R}\}$
 d) $\{x \mid x \geq -4, x \in \mathbb{R}\}$ or $\{x \mid x \leq -4, x \in \mathbb{R}\}$
 5. a) $f^{-1}(x) = \frac{1}{7}x$ b) $f^{-1}(x) = -\frac{1}{3}(x - 4)$
 c) $f^{-1}(x) = 3x - 4$ d) $f^{-1}(x) = 3x + 15$
 e) $f^{-1}(x) = -\frac{1}{2}(x - 5)$ f) $f^{-1}(x) = 2x - 6$

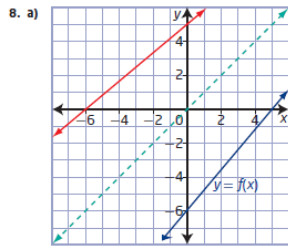
6. a) E b) C c) B d) A e) D



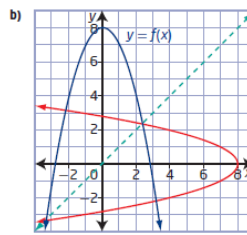
function: domain $\{-2, -1, 0, 1, 2\}$,
range $\{-2, 1, 4, 7, 10\}$
inverse: domain $\{-2, 1, 4, 7, 10\}$,
range $\{-2, -1, 0, 1, 2\}$



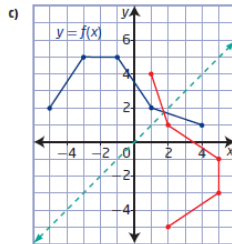
function: domain $\{-6, -4, -1, 2, 5\}$, range $\{2, 3, 4, 5\}$
inverse: domain $\{2, 3, 4, 5\}$, range $\{-6, -4, -1, 2, 5\}$



The inverse is a function; it passes the vertical line test.

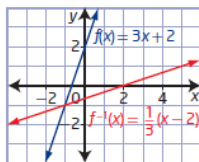


The inverse is not a function; it does not pass the vertical line test.



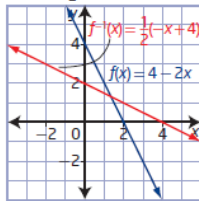
The inverse is not a function; it does not pass the vertical line test.

9. a) $f^{-1}(x) = \frac{1}{3}(x - 2)$



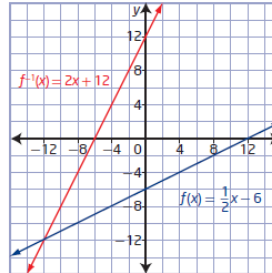
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

b) $f^{-1}(x) = \frac{1}{2}(-x + 4)$



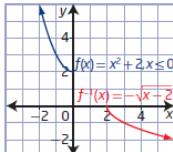
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

c) $f^{-1}(x) = 2x + 12$



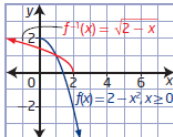
$f(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \in \mathbb{R}\}$, range $\{y \mid y \in \mathbb{R}\}$

d) $f^{-1}(x) = -\sqrt{x-2}$



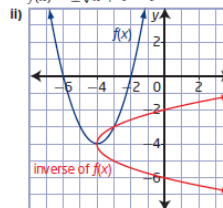
$f(x)$: domain $\{x \mid x \leq 0, x \in \mathbb{R}\}$, range $\{y \mid y \geq 2, y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \geq 2, x \in \mathbb{R}\}$, range $\{y \mid y \leq 0, y \in \mathbb{R}\}$

e) $f^{-1}(x) = \sqrt{2-x}$

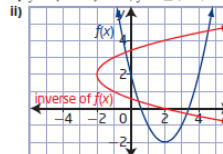


$f(x)$: domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$, range $\{y \mid y \leq 2, y \in \mathbb{R}\}$
 $f^{-1}(x)$: domain $\{x \mid x \leq 2, x \in \mathbb{R}\}$, range $\{y \mid y \geq 0, y \in \mathbb{R}\}$

10. a) i) $f(x) = (x + 4)^2 - 4$, inverse of $f(x) = \pm\sqrt{x+4} - 4$

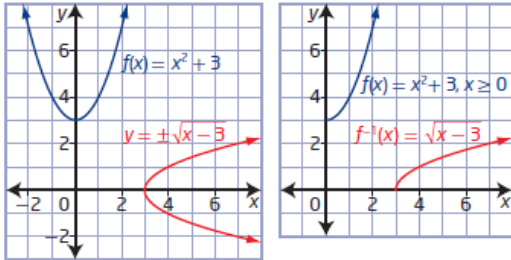


ii) $y = (x - 2)^2 - 2, y = \pm\sqrt{x+2} + 2$

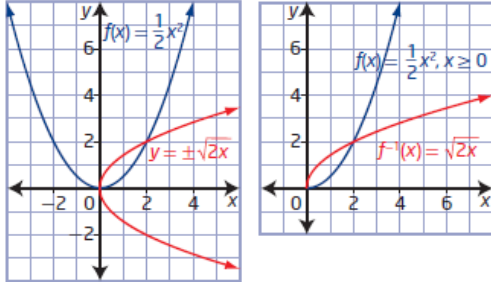


11. Yes, the graphs are reflections of each other in the line $y = x$.

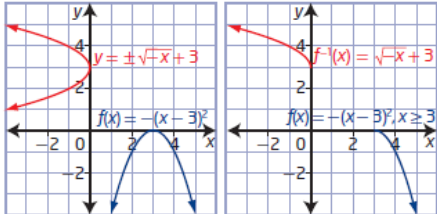
12. a) $y = \pm\sqrt{x-3}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



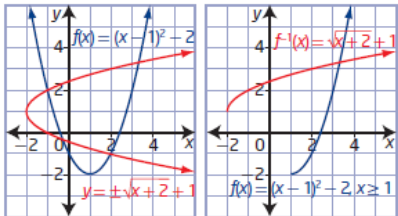
b) $y = \pm\sqrt{2x}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



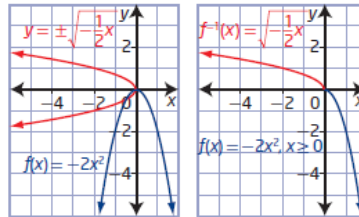
e) $y = \pm\sqrt{-x+3}$ restricted domain $\{x \mid x \geq 3, x \in \mathbb{R}\}$



f) $y = \pm\sqrt{x+2} + 1$ restricted domain $\{x \mid x \geq 1, x \in \mathbb{R}\}$



c) $y = \pm\sqrt{-\frac{1}{2}x}$ restricted domain $\{x \mid x \geq 0, x \in \mathbb{R}\}$



d) $y = \pm\sqrt{x-1}$ restricted domain $\{x \mid x \geq -1, x \in \mathbb{R}\}$

