Trinomial Decomposition

⑤ $8x^2 + 15x + 7 = 0$

$7 \times 8 = 56$

$7 + 8 = 15$

$(x + \frac{7}{8})(x + \frac{8}{8}) = 0$

$(8x + 7)(x + 1) = 0$

$8x + 7 = 0$ | $x + 1 = 0$

$8x = -7$ | $x = -1$

$x = -\frac{7}{8}$

⑥ $5x^2 + 11x + 6 = 0$

$5 \times 6 = 30$

$5 + 6 = 11$

$(x + \frac{5}{5})(x + \frac{6}{5}) = 0$

$(x + 1)(5x + 6) = 0$

$x + 1 = 0$ | $5x + 6 = 0$

$x = -1$ | $5x = -6$

$x = -\frac{6}{5}$

Examples:

Simple trinomial: $x^2 - x - 12 = 0$

$3 \times -4 = -12$

$3 + -4 = -1$

$(x + 3)(x - 4) = 0$

$x + 3 = 0$ | $x - 4 = 0$

$x = -3$ | $x = 4$

common factor: $5x^2 + 10x = 0$

$5x(x + 2) = 0$

$5x = 0$ | $x + 2 = 0$

$x = 0$ | $x = -2$

Diff of squares: $x^2 - 49 = 0$

$(x + 7)(x - 7) = 0$

$x + 7 = 0$ | $x - 7 = 0$

$x = -7$ | $x = 7$

⑧ $4n^2 - 16 = 0$

$a = 4 \quad b = 0 \quad c = -16$

by factoring:

$4n^2 - 16 = 0$

$4(n^2 - 4) = 0$

$4(n+2)(n-2) = 0$

$n + 2 = 0 \quad | \quad n - 2 = 0$

$n = -2 \quad | \quad n = 2$

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$x = \frac{-(0) \pm \sqrt{(0)^2 - 4(4)(-16)}}{2(4)}$$

$$x = \frac{0 \pm \sqrt{0 + 256}}{8}$$

$$x = \frac{0 \pm \sqrt{256}}{8}$$

$$x = \frac{0 \pm 16}{8}$$

$x = \frac{0+16}{8} = \frac{16}{8} = 2 \quad | \quad x = \frac{0-16}{8} = \frac{-16}{8} = -2$

$x = 2 \quad | \quad x = -2$

6.6 Vertex Form of a Quadratic Function

GOAL

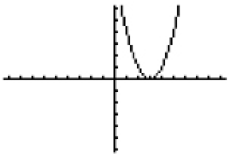
Graph a quadratic function in the form $y = a(x - h)^2 + k$, and relate the characteristics of the graph to its equation.

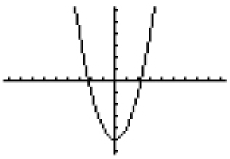
EXPLORE...

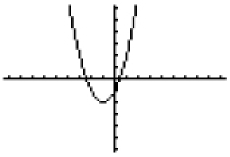
Quadratic functions can be written in different forms.
The basic quadratic function is $y = x^2$.
Use a calculator to graph the following quadratic functions.
Explain how the basic function is related to each function.
Describe how the changes in the function affect the graph.

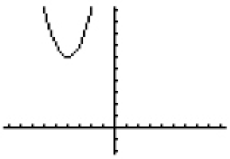
a) $y = (x - 3)^2$ → vertex: (3,0)
b) $y = x^2 - 5$ → vertex: (0,-5)
c) $y = (x + 1)^2 - 2$ → vertex: (-1,-2)
d) $y = (x + 4)^2 + 6$ → vertex: (-4,6)
e) $y = -2(x + 1)^2 + 3$ → vertex: (-1,3)
f) $y = 3(x - 2)^2 - 4$
→ (2,-4)

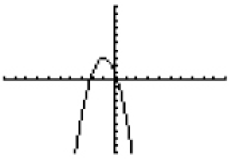
SAMPLE ANSWER

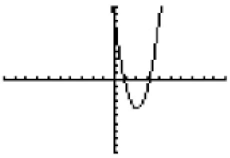
a)

Obtained from the basic function by subtracting 3 inside the brackets. This moves the graph of the basic function right 3 units.

b)

Obtained from the basic function by subtracting 5 outside the brackets. This moves the graph of the basic function down 5 units.

c)

Obtained from the basic function by adding 1 inside the brackets and subtracting 2 outside the brackets. This moves the graph of the basic function left 1 unit and down 2 units.

d)

Obtained from the basic function by adding 4 inside the brackets and adding 6 outside the brackets. This moves the graph of the basic function left 4 units and up 6 units.

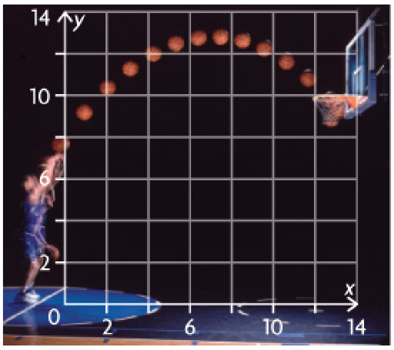
e)

Obtained from the basic function by adding 1 inside the brackets, and then multiplying by -2 and adding 3 outside the brackets. This moves the graph to the left by 1 unit and up by 3 units, flips the graph upside-down, and makes it narrower.

f)

Obtained from the basic function by subtracting 2 inside the brackets, and then multiplying by 3 and subtracting 4 outside the brackets. This moves the graph to the right by 2 units and down by 4 units and makes the graph narrower.

INVESTIGATE the Math

A high-school basketball coach brought in Judy, a trainer from one of the local college teams, to talk to the players about shot analysis. Judy demonstrated, using stroboscopic photographs, how shots can be analyzed and represented by quadratic functions. She used the following function to model a shot:

$$y = -0.1(x - 8)^2 + 13$$



$y = -0.1(x-8)^2 + 13$

vertex: (8, 13)

opens down ($a = -0.1$)

vertex is a max.

In this function, x represents the horizontal distance, in feet, of the ball from the player and y represents the vertical height, in feet, of the ball above the floor.

Judy mentioned that once she had a quadratic equation in this form, she did not need the photographs. She could quickly sketch a graph of the path of the ball just by looking at the equation.

? How could Judy predict what the graph of the quadratic function would look like?

A. Graph the following function:

$$y = x^2$$

Change the graph by changing the coefficient of x^2. Try both positive and negative values. How do the parabolas change as you change this coefficient?

B. For each function you graphed in part A, determine the coordinates of the vertex and the equation of the axis of symmetry.

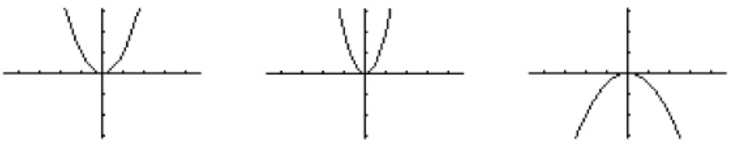
C. Graph this function:

$$y = x^2 + 1$$

Change the graph by changing the constant. Try both positive and negative values. How do the parabolas change as you change the constant? How do the coordinates of the vertex and the equation of the axis of symmetry change?

Answers

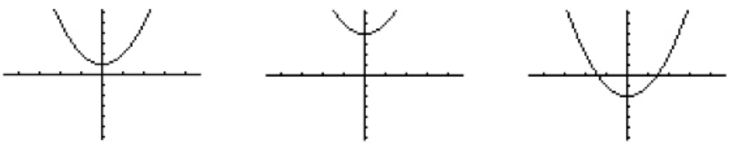
A.



Parabolas become narrower as the coefficient becomes more positive/negative; parabolas with negative coefficients open downward instead of upward.

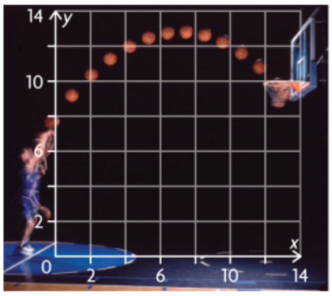
B. (0, 0), $x = 0$ (all functions)

C.



Parabolas shift up/down as the constant increases/decreases; the x-coordinate of the vertex is always 0, but the y-coordinate increases/decreases as the constant increases/decreases; the axis of symmetry is always $x = 0$.

INVESTIGATE the Math



A high-school basketball coach brought in Judy, a trainer from one of the local college teams, to talk to the players about shot analysis. Judy demonstrated, using stroboscopic photographs, how shots can be analyzed and represented by quadratic functions. She used the following function to model a shot:

$$y = -0.1(x - 8)^2 + 13$$

In this function, x represents the horizontal distance, in feet, of the ball from the player and y represents the vertical height, in feet, of the ball above the floor.

Judy mentioned that once she had a quadratic equation in this form, she did not need the photographs. She could quickly sketch a graph of the path of the ball just by looking at the equation.

? How could Judy predict what the graph of the quadratic function would look like?

D. Graph this function:

$$y = (x - 1)^2$$

Change the graph by changing the constant. Try both positive and negative values. How do the parabolas change as you change the constant? How do the coordinates of the vertex and the equation of the axis of symmetry change?

E. The equation that Judy used was expressed in vertex form:

$$y = a(x - h)^2 + k$$

Make a conjecture about how the values of a, h, and k determine the characteristics of a parabola.

F. Test your conjecture by predicting the characteristics of the graph of the following function:

$$y = -0.1(x - 8)^2 + 13$$

Use your predictions to sketch a graph of the function.

G. Using a graphing calculator, graph the function from part F:

$$y = -0.1(x - 8)^2 + 13$$

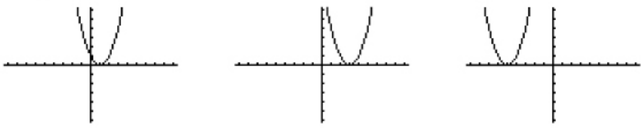
How does your sketch compare with this graph? Are your predictions supported? Explain.

Communication *Tip*

A quadratic function is in vertex form when it is written in the form $y = a(x - h)^2 + k$

Answers

D.

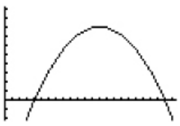


Parabolas shift right with a subtracted constant, left with an added constant; the y-coordinate of the vertex is always 0, but the x-coordinate increases/decreases as the (subtracted) constant increases/decreases; the axis of symmetry shifts with the x-coordinate of the vertex.

E. Conjecture: The vertex is (h, k) and the axis of symmetry is $x = h$; the parabola opens upward/downward for a positive/negative a; the parabola becomes narrower as a becomes more positive or negative.

F. Prediction: The vertex is (8, 13) and the axis of symmetry is $x = 8$; the parabola opens downward and is wider than the basic parabola $y = x^2$.

G.



The sketch has the same vertex, axis of symmetry, direction of opening, and general shape (wide) as a calculator graph; predictions are supported.

Reflecting

H. Does the value of a in a quadratic function always represent the same characteristic of the parabola, whether the function is written in standard form, factored form, or vertex form? Explain.

I. Neil claims that when you are given the vertex form of a quadratic function, you can determine the domain and range without having to graph the function. Do you agree or disagree? Explain.

J. Which form of the quadratic function—standard, factored, or vertex—would you prefer to start with, if you wanted to sketch the graph of the function? Explain.

Answers

H. Yes. It always represents the direction of opening and narrowness of the parabola.

I. Agree. If I were given $f(x) = 2(x + 3)^2 + 5$, I would know that the graph opens upward because 2 is positive, and that the vertex would be located at $(-3, 5)$. This means that the function contains a minimum value at the vertex. The domain would be $\{x \in R\}$ and the range would be $\{y \geq 5, y \in R\}$.

Disagree. If the context of the problem were a "projectile" question, then I would need to know the x-intercepts to state the domain, and I would not be able to state them simply by looking at the equation in vertex form.

J. Standard form would be fine for graphing the function with technology, and would be useful if I needed the y-intercept. I would prefer factored form if I wanted to locate the x-intercepts exactly. I would prefer vertex form if I was most interested in locating the vertex.

APPLY the Math

EXAMPLE 1 Sketching the graph of a quadratic function given in vertex form

Sketch the graph of the following function:

$$f(x) = 2(x - 3)^2 - 4$$

State the domain and range of the function.

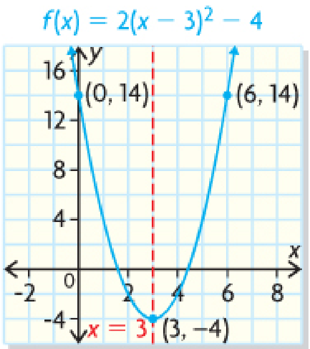
Samuel's Solution

$f(x) = 2(x - 3)^2 - 4$
Since $a > 0$, the parabola opens upward.
The vertex is at $(3, -4)$.
The equation of the axis of symmetry is $x = 3$.

The function was given in vertex form. I listed the characteristics of the parabola that I could determine from the equation.

$f(0) = 2(0 - 3)^2 - 4$
$f(0) = 2(-3)^2 - 4$
$f(0) = 2(9) - 4$
$f(0) = 18 - 4$
$f(0) = 14$
Point $(0, 14)$ is on the parabola.

To determine another point on the parabola, I substituted 0 for x.



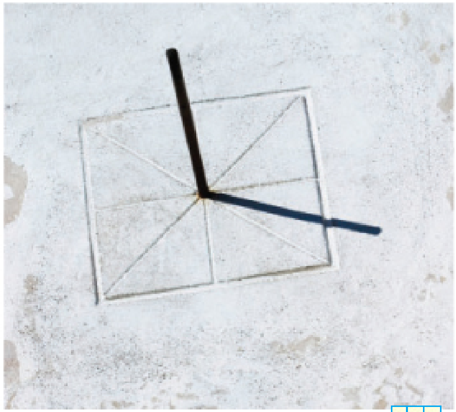
I plotted the vertex and the point I had determined, (0, 14). Then I drew the axis of symmetry. I used symmetry to determine the point that is the same horizontal distance from (0, 14) to the axis of symmetry. This point is (6, 14). I connected all three points with a smooth curve.

Domain and range:
$\{(x, y) \mid x \in \mathrm{R}, y \geq -4, y \in \mathrm{R}\}$

EXAMPLE 2 Determining the equation of a parabola using its graph

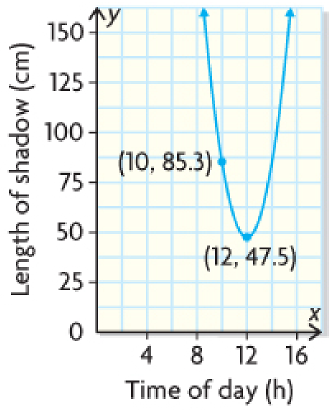
Liam measured the length of the shadow that was cast by a metre stick at 10 a.m. and at noon near his home in Saskatoon. Other students in his class also measured the shadow at different times during the day. They had read that, when graphed as shadow length versus time, the data should form a parabola with a minimum at noon, because the shadow is shortest at noon. Liam decided to try to predict the equation of the parabola, without the other students' data.

Determine the equation that represents the relationship between the time of day and the length of the shadow cast by a metre stick.



Liam's Solution

I have the points (10, 85.3) and (12, 47.5).



I measured the length of the shadow in centimetres. My measurements were 85.3 cm at 10 a.m. and 47.5 cm at noon.

I drew a sketch of a parabola using (12, 47.5) as the vertex, since the length of the shadow at noon should be the minimum value of the function.

$f(x) = a(x - h)^2 + k$
$f(x) = a(x - 12)^2 + 47.5$

I decided to use the vertex form of the quadratic function, since I already knew the values of *h* and *k* in this form.

Solving for *a*:
$85.3 = a(10 - 12)^2 + 47.5$
$85.3 = a(-2)^2 + 47.5$
$85.3 = 4a + 47.5$
$37.8 = 4a$
$9.45 = a$

I knew that (10, 85.3) is a point on the parabola. I substituted the coordinates of this point into the equation and then solved for *a*.

The function that represents the parabola is
$f(x) = 9.45(x - 12)^2 + 47.5$

The domain and range of this function depend on the hours of daylight, which depends on the time of year.

EXAMPLE 3 Reasoning about the number of zeros that a quadratic function will have

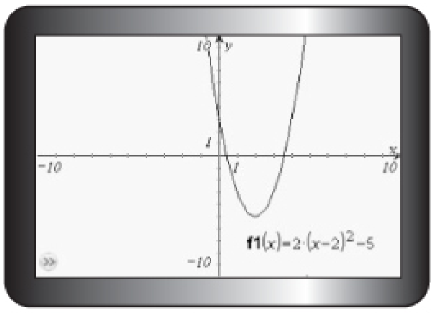
Randy claims that he can predict whether a quadratic function will have zero, one, or two zeros if the function is expressed in vertex form. How can you show that he is correct?

TI-*nspire*

Eugene's Solution

$f(x) = 2(x - 2)^2 - 5$
Conjecture: two zeros

The vertex of the parabola that is defined by the function is at $(2, -5)$, so the vertex is below the x-axis. The parabola must open upward because a is positive. Therefore, I should observe two x-intercepts when I graph the function.



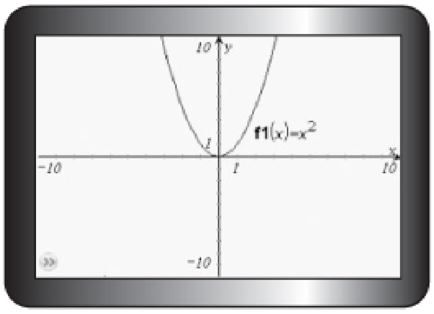
To test my conjecture, I graphed the function on a calculator. I can see two x-intercepts on my graph, so the function has two zeros.

The graph supports my conjecture.

$f(x) = x^2$
$f(x) = (x - 0)^2 + 0$
Conjecture: one zero

I decided to use the basic quadratic function, since this provided me with a convenient location for the vertex, $(0, 0)$.

Since the vertex is on the x-axis and the parabola opens up, this means that I should observe only one x-intercept when I graph the function.

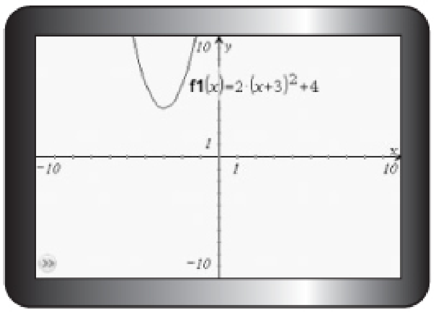


To test my conjecture, I graphed the function on a calculator. Based on my graph, I concluded that the function has only one zero.

The graph supports my conjecture.

$f(x) = 2(x + 3)^2 + 4$
Conjecture: no zeros

The vertex of the parabola that is defined by this function is at $(-3, 4)$, and the parabola opens upward. The vertex lies above the x-axis, so I should observe no x-intercepts when I graph the function.



To test my conjecture, I graphed the function on a calculator. I concluded that the function has no zeros.

The graph supports my conjecture.

EXAMPLE 4 Solving a problem that can be modelled by a quadratic function

A soccer ball is kicked from the ground. After 2 s, the ball reaches its maximum height of 20 m. It lands on the ground at 4 s.

a) Determine the quadratic function that models the height of the kick.

b) Determine any restrictions that must be placed on the domain and range of the function.

c) What was the height of the ball at 1 s? When was the ball at the same height on the way down?

Tia's Solution

a) Let x represent the elapsed time in seconds, and let y represent the height in metres.

$y = a(x - h)^2 + k$

Since I knew the maximum height and when it occurred, I also knew the coordinates of the vertex. I decided to use the vertex form to determine the equation.

The maximum height is 20 m at the elapsed time of 2 s.

Vertex:

$(x, y) = (2, 20)$

$y = a(x - 2)^2 + 20$

I substituted the known values.

Solving for a:

$f(x) = a(4 - 2)^2 + 20$

$0 = a(2)^2 + 20$

$0 = 4a + 20$

$-20 = 4a$

$-5 = a$

To determine the value of a, I substituted the coordinates of the point that corresponds to the ball hitting the ground, (4, 0).

The following quadratic function models the height of the kick:

$f(x) = -5(x - 2)^2 + 20$

b) Time at beginning of kick:

$x = 0$

Time when ball hits ground:

$x = 4$

Domain: $\{x \mid 0 \leq x \leq 4, x \in R\}$

At the beginning of the kick, the time is 0 s. When the ball lands, the time is 4 s. I can only use x-values in this interval. Time in seconds is continuous, so the set is real numbers.

Vertex: (2, 20)

Height of ball at beginning of kick: 0 m

Height of ball at vertex: 20 m

Range: $\{y \mid 0 \leq y \leq 20, y \in R\}$

The ball starts on the ground, at a height of 0 m, and rises to its greatest height, 20 m. The ball is not below the ground at any point. Height in metres is continuous, so the set is real numbers.

c) $f(x) = -5(x - 2)^2 + 20$

$f(1) = -5(1 - 2)^2 + 20$

$f(1) = -5(-1)^2 + 20$

$f(1) = -5 + 20$

$f(1) = 15$

I used the vertex form of the quadratic function to determine the height of the ball at 1 s.

The ball was at a height of 15 m after 1 s. This occurred as the ball was rising.

Equation of the axis of symmetry:

$x = 2$

Symmetry provides the point (3, 15). The ball was also 15 m above the ground at 3 s. This occurred as the ball was on its way down.

I knew that point (1, 15) is 1 unit to the left of the axis of symmetry of the parabola. The other point on the parabola, with height 15 m, should be 1 unit to the right of the axis of symmetry. This means that the x-coordinate of the point must be 3.

In Summary

Key Idea

- The vertex form of the equation of a quadratic function is written as follows:

$$y = a(x - h)^2 + k$$

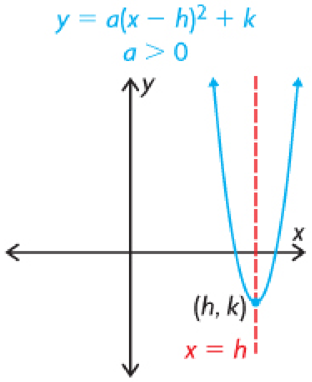
The graph of the function can be sketched more easily using this form.

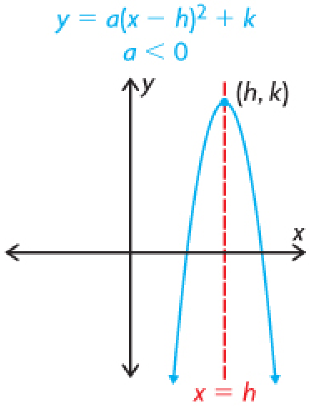
Need to Know

- A quadratic function that is written in vertex form,

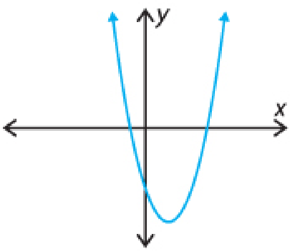
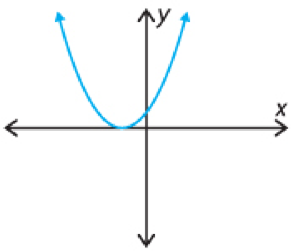
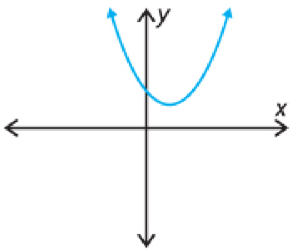
$$y = a(x - h)^2 + k$$

has the following characteristics:
  - The vertex of the parabola has the coordinates (h, k).
  - The equation of the axis of symmetry of the parabola is $x = h$.
  - The parabola opens upward when $a > 0$, and the function has a minimum value of k when $x = h$.
  - The parabola opens downward when $a < 0$, and the function has a maximum value of k when $x = h$.





- A parabola may have zero, one, or two x-intercepts, depending on the location of the vertex and the direction in which the parabola opens. By examining the vertex form of the quadratic function, it is possible to determine the number of zeros, and therefore the number of x-intercepts.

Two x-intercepts	One x-intercept	No x-intercepts
		

Assignment: pages 335 - 337

Questions 1, 2(ac), 3, 4, 5, 8

SOLUTIONS => 6.6 Vertex Form of a Quadratic Function.

1. For each quadratic function below, identify the following:

i) the direction in which the parabola opens.
ii) the coordinates of the vertex
iii) the equation of the axis of symmetry.

a) $f(x) = (x-3)^2 + 7$

i) Opens Upward
ii) Vertex: $(3, 7)$
iii) Axis of Symmetry: $x = 3$

b) $m(x) = -2(x+7)^2 - 3$.

i) Opens Downward
ii) Vertex $(-7, -3)$
iii) Axis of Symmetry: $x = -7$

c) $g(x) = 7(x-2)^2 - 9$

i) Opens Upward
ii) Vertex $(2, -9)$
iii) Axis of Symmetry: $x = 2$

d) $n(x) = \frac{1}{2}(x+1)^2 + 10$

i) Opens Upward
ii) Vertex $(-1, 10)$
iii) Axis of Symmetry: $x = -1$

e) $r(x) = -2x^2 + 5$

$r(x) = -2(x-0)^2 + 5$

i) Opens Downward

ii) Vertex (0,5)

iii) Axis of Symmetry: $x = 0$

2. Predict which of the following functions have a minimum value. Also predict the number of x-intercepts that each function has. Test your predictions by sketching the graph of each function.

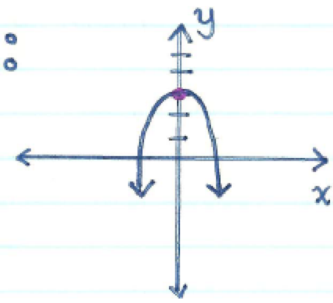
{a & c}

a) $f(x) = -x^2 + 3$

$f(x) = -(x-0)^2 + 3$

Vertex: $(0,3)$

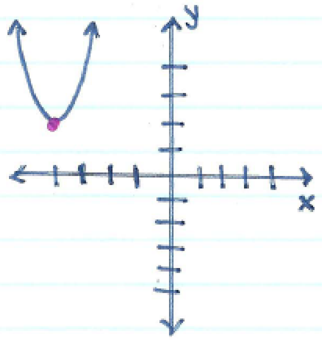
Sketch:



↳ Opens Downward => Maximum

↳ 2 x-intercepts

Sketch:



c) $m(x) = (x+4)^2 + 2$

Vertex: $(-4,2)$

↳ Opens Upward => Minimum

↳ No x-intercepts

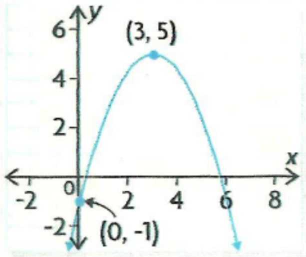
3. Determine the value of a, if point (-1,4) is on the quadratic function:

$$f(x) = a(x+2)^2 + 7$$

$(-1, 4)$
$x \quad y$

$$y = a(x+2)^2 + 7$$
$$4 = a(-1+2)^2 + 7$$
$$4 = a(1)^2 + 7$$
$$4 = a(1) + 7$$
$$4 - 7 = 1a$$
$$\frac{-3}{1} = \frac{1a}{1}$$
$$-3 = a$$

4. Which equation represents the graph? Justify your decision.



* Opens Downward

** Vertex (3,5)

*** $c = -1$ (0,-1)

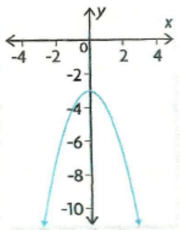
A. $y = -\frac{2}{3}x^2 + 5$

$\boxed{\text{C.}}$ $y = -\frac{2}{3}(x-3)^2 + 5$

B. $y = -(x-3)^2 + 5$

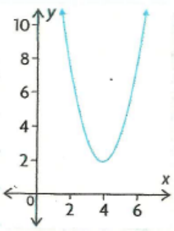
D. $y = \frac{2}{3}(x-3)^2 + 5$

5. Match each equation with its corresponding graph. Explain your reasoning.

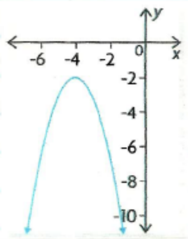
i) 

Match: c) $y = -x^2 - 3$

Vertex $(0, -3)$; Opens Downward

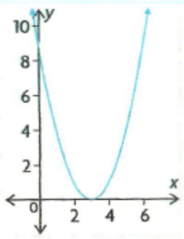
ii) 

Match: d) $y = (x-4)^2 + 2$

iii) 

Match: b) $-(x+4)^2-2$

Vertex $(-4,-2)$; Opens Downward

iv) 

Match: a) $y=(x-3)^2$

Vertex $(3,0)$: Opens Upward

8. Marleen and Candice are both 6 ft tall, and they play on the same college volleyball team. In a game, Candice set up Marleen with an outside high ball for an attack hit. Using a video of the game, their coach determined that the height of the ball above the court, in feet, on its path from Candice to Marleen could be defined by the function

$$h(x) = -0.03(x-9)^2 + 8$$

where x is the horizontal distance, measured in feet, from one edge of the court.

a) Determine the axis of symmetry of the parabola.

Vertex (9, 8) Axis of Symmetry $\boxed{x=9}$

b) Marleen hit the ball at its highest point. How high above the court was the ball when she hit it?

The ball was 8 ft above the court when she hit it.

c) How high was the ball when Candice set it, if she was 2 ft from the edge of the court?

$h = -0.03(x-9)^2 + 8$
$h = -0.03(2-9)^2 + 8$
$h = -0.03(-7)^2 + 8$
$h = -0.03(49) + 8$
$h = -1.47 + 8$
$h = 6.53$ ft

The ball was 6.53 ft high when Candice set it.

d) State the range for the ball's path between Candice and Marleen.

$\{h \mid 6.5 \leq h \leq 8, h \in R\}$

Attachments

7s6e1 final.mp4

7s6e2 final.mp4

7s6e3 final.mp4

7s6e4 final.mp4

fm7s6-p9.tns

FM11-7s6-ahk.gsp

FM11-7s6.gsp